# Pension Design in the Presence of Systemic Risk 

Stavros Panageas*<br>University of Chicago - Booth School of Business and NBER

January 2013


#### Abstract

Individual agents' savings and portfolio choices can have systemic, negative externalities on public finances when a minimum level of retirement consumption is not assured. I discuss optimal policies that prevent such an outcome. Specifically, I show the optimality of two policies funded by optimally determined mandatory savings. The first policy mandates the use of accumulated savings to purchase a claim providing a fixed income stream during retirement. The second policy mandates an appropriately structured portfolio insurance policy. It is also shown that borrowing constraints make it optimal to "backload" mandatory savings towards the end of an agent's work-life.


Keywords: Continuous time optimization, Life Cycle savings and portfolio choice, Portfolio insurance, Ricardian Equivalence, Borrowing constraints

JEL Classification: C6, D6, D9, E2, E6, G1

[^0]
## 1 Introduction

This paper develops a methodology to design optimal, fully funded claims aimed at influencing consumers' optimal savings and portfolio choices. The motivation for influencing these choices lies in negative externalities on public finances, which arise when a consumer's post-retirement standard of living drops below a minimum level. In the absence of any constraints, other than the intertemporal budget constraint, such fully funded claims do not exist. ${ }^{1}$ However, in the presence of borrowing constraints one can explicitly determine optimal claims to ensure a minimum, post-retirement standard of living.

The economic context is provided by recent trends in retirement economics. Specifically, the world-wide trend towards fully funded, personal retirement accounts has raised concerns that some retirees may arrive in retirement with insufficient funds and diminished precautions against a stock market downturn. In the presence of redistribution mechanisms, such an event could create pressures to provide direct or indirect transfers to the affected retirees, increasing distortionary taxes, and having a "systemic" impact on public finances. ${ }^{2}$

Because of these concerns, it is very common for countries to complement the shift towards private accounts and defined contribution plans with various measures to ensure a minimum standard of living in retirement. Such measures include minimum return guarantees, minimum retirement incomes, phased (as opposed to lump sum) withdrawals upon entering retirement, and mandates to use part of the accumulated balances to purchase a fixed annuity and ensure a minimum defined benefit. ${ }^{3}$ The idea to use similar measures is also the topic of policy discussion in the US. ${ }^{4}$

[^1]The pervasive use of measures to ensure a minimum standard of living in retirement has led to various studies evaluating the costs and benefits of specific policy interventions adopted in certain countries. ${ }^{5}$ Less emphasis has been placed on developing an integrated methodology to derive the optimal government policies that would ensure a minimum standard of living in a fully funded retirement system. The present paper takes a step in that direction. It proposes a framework to formalize the issues discussed above, and then develops a new methodology to solve for the optimal provision of retirement benefits. This new methodology utilizes approaches developed in the last two decades primarily in financial economics, and particularly in the strands of the literature analyzing portfolio insurance problems.

The proposed framework features a benevolent, rational government aiming to maximize social welfare. Agents in the society maximize their individual welfare, which does not coincide with social welfare. The reason for the discrepancy is that a representative agent's consumption in retirement can have negative, external effects. This occurs when retirement consumption drops below a given minimum level and triggers redistributive pressures financed by distortionary taxes on the population.

To avoid such negative external effects, the government optimally designs policies to ensure that a retiree's consumption does not fall below the specified minimum level. The allowed government policies are transfers from and to the agent. They can be chosen subject to two constraints:

The first constraint is informational, and reflects an intentionally conservative assumption on the government's information set. Specifically, the government can condition its policies on aggregate outcomes (e.g., the return on the stock market), but not on individual consumption, savings and portfolio choices. Since the government cannot dictate consumption, savings and portfolio choices, it needs to induce the agent to choose specific consumption and portfolio paths.

[^2]The second constraint is a full-financing constraint. The net present value of the transfers provided to the agent should be equal to the present value of mandatory savings accumulated by the agent. This assumption is in line with the aim of the paper, which is to discuss optimal policies ensuring a minimum standard of living, assuming that society has opted for a fully funded system. (The full funding assumption also helps isolate the tensions that arise between a fully funded retirement system and a redistributive welfare system aimed to insure intra-generational risks in retirement. ${ }^{6}$ ) The broader issue of the advantages and disadvantages of full funding - as opposed to "pay as you go" - is outside the scope of this paper, and the reader is referred to the large literature that discusses this issue. ${ }^{7,8}$

These two assumptions are made for two reasons: a) to be as conservative as possible about the government's ability to observe and influence agents' actions, ${ }^{9}$ and b) in order to make the theoretical results of the paper more interesting. To elaborate on the second reason, it is useful to recall that in the absence of frictions, only the present value of an agent's resources restricts her consumption choices. ${ }^{10}$ Hence, according to the joint assumptions that a) individual choices cannot be mandated, and b) the transfers received by the agent are financed by herself during her work-years, it would seem that no government intervention can succeed in affecting the agent's consumption choices.

To allow governmental policies to have a meaningful effect, I make a third assumption,

[^3]namely that agents cannot borrow against future governmental transfers. (Such constraints can be easily enforced in courts by forbidding securitization of such payments). Because of the resulting borrowing constraint, the government can affect the agent's consumption choices and it becomes possible to discuss optimal mandatory savings and transfer processes.

The main contribution of the paper is to develop a methodology to design an optimal preretirement mandatory savings/ and post-retirement transfer process that exploits borrowing constraints, so as to induce savings and consumption choices consistent with a minimum standard of living in retirement. The proposed methodology is based on the literature on convex duality / dynamic Lagrange multiplier methods (Basak and Cuoco (1998), Chien et al. (2007), Cuoco (1997), Cvitanic and Karatzas (1992), Detemple and Serrat (2003), Dumas and Lyasoff (2010), Gallmeyer and Hollifield (2008), He and Pages (1993), He and Pearson (1991), Haugh et al. (2004), Lustig (2002), Marcet and Marimon (1998)). ${ }^{11}$ The typical approach in this literature is to take a (post-transfer) income process of the agent as given and derive the process of Lagrange multipliers associated with the borrowing constraint. The new methodological aspect of the present paper is that the convex duality approach is applied in reverse. The government first solves for the optimal consumption process, derives the associated optimal process for the Lagrange multipliers, and then determines a transfer process that is associated with these Lagrange multipliers.

Utilising this methodology, I show that there can be multiple optimal forms of pension design. Somewhat surprisingly - given the wide variety of admissible policies - one optimal policy takes the form of a simple, fixed, mandatory annuity: under such a policy agents are required to use a fraction of their assets upon entering retirement in order to purchase a fixed income stream for the duration of their life. The level of that fixed income stream is explicitly derived and shown to be a multiple of the minimum level of consumption that the government is aiming to enforce. The optimality of the fixed annuity critically hinges on the

[^4]fact the investment opportunity set (interest rate, market price of risk) is constant. For an arbitrary investment opportunity set, I derive an alternative optimal "portfolio insurance" policy, which delivers appropriate transfers once the value of the agent's portfolio borders on zero. Both transfer processes are optimally financed by mandatory savings that are accumulated during the latter years of an agent's worklife, when borrowing constraints have ceased binding.

In terms of substance, the paper relates to two strands of the literature.
The first strand is the finance literature on portfolio insurance. (See, e.g., Basak (2002), Grossman and Zhou (1996).) In that literature some agents voluntarily place a requirement on the minimum level of their assets at some point in time. In the present paper agents must be induced to adopt consumption and asset accumulation plans that can safeguard such a minimum standard of living, otherwise they can cause negative externalities.

The second strand is the literature on "dynamic public finance". ${ }^{12}$ That literature considers optimal insurance and contract design problems predominantly in setups of "hidden information" due to idiosyncratic shocks with or without observable savings. ${ }^{13}$ The present paper differs from that literature in its scope. The main friction assumed in this paper is that agents' incentives to hedge against downturns in their consumption - caused by a combination of portfolio/savings decisions and common, aggregate shocks - may be inadequate from the perspective of a central planner. To illustrate the novel intuitions obtained in such a framework, most of the analysis focuses exclusively on common, aggregate shocks and ab-

[^5]stracts from idiosyncratic shocks. (However, unobserved idiosyncratic shocks are considered in the appendix to the paper as a foundation for the presence of a redistributive welfare system). To be clear, this focus on aggregate shocks is not meant to deny or downplay the obvious importance of idiosyncratic shocks in the real world; it is however a useful theoretical abstraction, so as to isolate the paper's new predictions.

The paper is structured as follows. Section 2 sets up the model. Section 3 introduces a government with the task of keeping the agent's consumption above a minimum level by usage of appropriate fully funded transfers. Section 4 considers the agent's reaction to the presence of such intervention. Section 5 derives an upper bound to welfare no matter which set of admissible taxes/transfers is utilized. Section 6 illustrates two distinct ways of attaining that upper bound, which are hence optimal. Section 7 discusses pre-retirement implications and mandatory savings. Section 8 discusses the implications of closing the model in general equilibrium. Section 9 concludes. Appendix A provides further details on the assumed negative externality that arises when an agent's consumption drops below the minimum level. All proofs are in appendix B.

## 2 The model

### 2.1 Agents, preferences, and endowments

The baseline model is very similar to the small open economy version of Blanchard (1985). Accordingly, the investment opportunity set (interest rate, equity premium etc.) is taken as given. Section 8 shows that the important results of this baseline model remain valid in a closed, general-equilibrium economy with endogenous interest rates and equity premia.

All agents are identical. The typical agent faces a probability of death $q$ per unit of time $d t$. All agents have constant relative risk aversion $\gamma$, and a constant discount rate $\rho$.

Accordingly, the maximization problem of an agent, who is born at time $t^{b}$, is given by

$$
\begin{equation*}
E_{t^{b}} \int_{t^{b}}^{\infty} e^{-(\rho+q)\left(t-t^{b}\right)} \frac{c_{t}^{1-\gamma}}{1-\gamma} d t \tag{1}
\end{equation*}
$$

To expedite the exposition and shorten proofs, I concentrate on the empirically relevant case $\gamma>1 .{ }^{14}$

Life has two phases. A "work" phase, which lasts for $\tau$ years after birth, and is followed by a "retirement phase". During the work phase agents receive a constant income stream equal to $Y$ per unit of time. Once they retire, they receive no more labor income.

### 2.2 Investment opportunity set

Agents can invest in the money market, where they receive a constant strictly positive interest rate $r>0$. In addition, they can invest in a risky security with a price-per-share process

$$
\begin{equation*}
\frac{d P_{t}}{P_{t}}=\mu d t+\sigma d B_{t} \tag{2}
\end{equation*}
$$

where $\mu>r$ and $\sigma>0$ are given constants and $B_{t}$ is a one-dimensional Brownian motion on a complete probability space $(\Omega, F, P) .{ }^{15}$ The realization of this Brownian motion is the only source of uncertainty in this economy. The extension to multiple assets is straightforward and is left out.

As is well understood, dynamic trading in the stock and the bond implies a dynamically complete market. (See, e.g., Duffie (2001) or Karatzas and Shreve (1998)). Specifically, there exists a unique stochastic discount factor $H_{t}$, so that the time- $t$ price of any claim that

[^6]delivers dividends equal to $D_{u}$, for $u \geq t$ is given by
$$
E_{t} \int_{t}^{\infty} \frac{H_{u}}{H_{t}} D_{u} d u
$$

As Karatzas and Shreve (1998) show, the stochastic discount factor that is consistent with a constant interest rate and the price-per-share process (2) is given by

$$
\begin{equation*}
\frac{d H_{t}}{H_{t}}=-r d t-\kappa d B_{t}, \text { where } \kappa \equiv \frac{\mu-r}{\sigma} . \tag{3}
\end{equation*}
$$

The agent can also enter into "annuity-style" contracts with a competitive life insurance company as in Blanchard (1985). Specifically, these contracts specify the following cashflows: The insurance company offers an income stream of $p$ per unit of time $d t$, in exchange for receiving one dollar if the agent dies over the next interval $d t$. Competition between insurance companies implies that $p=q$. The presence of such annuities is inessential for the main arguments, but it simplifies some technical aspects of the analysis.

### 2.3 Portfolio and wealth processes

Throughout life, an agent chooses a portfolio process $\pi_{t}$ and a consumption process $c_{t}$. The portfolio process $\pi_{t}$ is the dollar amount invested in the risky asset (the "stock market") at time $t$. The rest, $W_{t}-\pi_{t}$, is invested in the money market. Since the key insights of the paper do not depend on the presence or absence of bequest motives, I simplify matters and assume that the agent has no bequest motives. ${ }^{16}$ As a result, the agent has an incentive to enter Blanchard-style annuity contracts for the full amount of her financial wealth. This results in an income stream of $q W_{t}$ per unit of time $d t$ while she is alive. Accordingly, the

[^7]wealth process of a retired agent evolves as
\[

$$
\begin{equation*}
d W_{t}=q W_{t} d t+\pi_{t}\left\{\mu d t+\sigma d B_{t}\right\}+\left\{W_{t}-\pi_{t}\right\} r d t-c_{t} d t \tag{4}
\end{equation*}
$$

\]

and the wealth process of a working agent is given by:

$$
\begin{equation*}
d W_{t}=q W_{t} d t+\pi_{t}\left\{\mu d t+\sigma d B_{t}\right\}+\left\{W_{t}-\pi_{t}\right\} r d t+Y d t-c_{t} d t . \tag{5}
\end{equation*}
$$

An additional requirement is that financial wealth must remain non-negative throughout:

$$
\begin{equation*}
W_{t} \geq 0 \text { for all } t \tag{6}
\end{equation*}
$$

This constraint excludes un-collateralized borrowing. However, collateralized borrowing (using the stock as collateral to borrow bonds and vice versa) is allowed. Alternatively phrased, the agent can enter negative positions in the bond (resp. stock) market, as long as the sum of the value of her bond and stock holdings is non-negative.

### 2.4 Externalities when consumption falls below a minimum standard of living

As already mentioned in the introductory section, almost all societies that rely on funded retirement systems typically complement them with regulatory measures to ensure that retiree consumption does not fall below a minimum standard.

The motivations for such regulatory interventions fall in two broad categories: behavioral and rational. According to the behavioral perspective, even though individuals may recognize the presence of some inelastic expenditures in retirement (health care and nursing costs etc.), they may go through life without making provisions for such a minimum post-retirement standard of living. According to the rational perspective, the rationale for intervention
is different: If agents find themselves in situations where their consumption falls below a minimum level, they may demand redistributive transfers, imposing a cost on taxpayers.

Even though the rational and behavioral perspectives on regulatory intervention are diametrically different, they coincide in one important aspect. According to both views, a drop in retiree consumption below some minimum level $\xi$ would be associated with adverse effects, which are external to the decision-making agent.

Specifically, the behavioral view formalizes this notion by postulating an agent with two "selves", a "prudent" self and an "imprudent" self. The "prudent" self correctly perceives that if her post-retirement consumption falls below a minimum level required for nursing, health care etc., this would be associated with a very large disutility (say negative infinity). Accordingly, the prudent self's preference is to maximize (1) subject to the constraint:

$$
\begin{equation*}
c_{t} \geq \xi \text { for all } t>t^{b}+\tau \tag{7}
\end{equation*}
$$

However, the imprudent self, who is assumed to be the decision-making agent, maximizes (1) without regard to to the constraint (7). Because of this divergence of objectives, the actions of the decision-making, imprudent self have external effects on the welfare of the nonacting, prudent self. This introduces a role for government intervention: The government is assumed to maximize the welfare of the prudent self, while taking into account that decisions will be made by the imprudent self. The literature models such situations by employing the framework of principal-agent problems of the sort that will be presented in the next section. ${ }^{17}$

The rational perspective to governmental intervention refutes the idea that agents have multiple selves. Instead, a consumption drop below the minimum level $\xi$ is assumed to have an adverse impact on society. Appendix A illustrates the source of such an externality by presenting a stylized, extended model featuring a tax-financed welfare system, intended for

[^8]agents experiencing unobserved catastrophic shocks in retirement. In the context of that model, an agent with consumption below $\xi$ would find it worthwhile to exert effort and falsely claim that she has experienced a catastrophic shock, so that she can collect taxfinanced transfers, which impose deadweight costs on society. As a result constraint (7) arises endogenously as a "truth telling" constraint, preventing the associated deadweight costs. To expedite the presentation of the main results, I present the details of such a model in the appendix; the body of the paper simply assumes that a violation of constraint (7) would lead to substantive externalities on society, which the central planner wants to avoid.

To summarize, both behavioral and rational perspectives share some common ground. Under both perspectives, the government should strive to ensure that agents' decisions satisfy the constraint (7), while recognizing that decision-making agents would not subject their decisions to the requirements of (7) if left alone. This common ground allows a formulation of the government's objective in the next section, without having to differentiate between rational and behavioral views.

## 3 Introducing a role for the government

To achieve the goal of imposing constraint (7) on the agent's choices, the government can use transfers to modify the agent's behavior so that retirees' consumption plans satisfy equation (7). The government can observe an agent's income and the realized returns on the stock market, but not the agent's assets or her consumption.

Based on that information set, the government needs to structure fully funded transfers to the individual so as to ensure that constraint (7) holds. To keep with the assumption that the retirement system is fully funded, such transfers are financed by the agent upon entering retirement.

To obtain these optimal transfers it is most useful to use backward induction and split the problem into a "post-retirement" part (which is solved first) and a "pre-retirement"
part, which is solved subsequently. In the post-retirement part the government determines the optimal transfer process that maximizes the agent's retirement utility subject to (7), assuming that these transfers are financed with an upfront payment upon entering retirement. This is done in sections 3.1-6. The pre-retirement part is discussed in section 7 .

### 3.1 The post-retirement problem

Because of the time-invariance of the problem, I henceforth simplify notation and normalize the time of retirement $t^{b}+\tau$ to be equal to zero. For the analysis of the post-retirement problem (sections 3.1-6), I also normalize the value of the stochastic discount factor at retirement to be equal to $H_{0}=1 .{ }^{18}$

Problem 1 The government's objective is to determine a cumulative non-decreasing transfer process $G_{t}$ and an initial tax $D_{0}$ so as to maximize:

$$
\begin{equation*}
\Omega\left(W_{0}\right) \equiv \max _{G_{t}, D_{0}} E_{0} \int_{0}^{\infty} e^{-(\rho+q) t} \frac{c_{t}^{1-\gamma}}{1-\gamma} d t \tag{8}
\end{equation*}
$$

subject to

$$
\begin{align*}
c_{t} & \geq \xi \text { for all } t>0,  \tag{9}\\
D_{0} & =E_{0} \int_{0}^{\infty} e^{-q t} H_{t} d G_{t} \tag{10}
\end{align*}
$$

and subject to the constraint that $c_{t}$ solves the decision-making agent's optimization problem

[^9]given $G_{t}$
\[

$$
\begin{align*}
& c_{t}=\arg \max _{\left.<c_{t}, \pi_{t}\right\rangle} E_{0} \int_{0}^{\infty} e^{-(\rho+q) t} \frac{c_{t}^{1-\gamma}}{1-\gamma} d t  \tag{11}\\
& \text { s.t. } \\
& d W_{t}=q W_{t} d t+\pi_{t}\left\{\mu d t+\sigma d B_{t}\right\}+\left\{W_{t}-\pi_{t}\right\} r d t-c_{t} d t+d G_{t}  \tag{12}\\
& W_{0^{+}}=W_{0}-D_{0}  \tag{13}\\
& W_{t} \geq 0 \text { for all } t>0 \tag{14}
\end{align*}
$$
\]

Consistent with the behavioral and rational motivations given previously, the government aims to maximize (8), subject to the additional requirement (9) that the agent's consumption not fall below the minimum level $\xi$.

Equations (10) and (13) state that the cost of providing the transfer process $G_{t}$ to the consumer should be self-financed by an upfront payment $D_{0}$. Parenthetically, this "selffinancing" requirement implies that the government is not required to implement the provision of transfers to consumers. It can simply specify the optimal process $G_{t}$ that each consumer should purchase and leave it to competitive financial companies to price and provide these transfers.

Finally, equations (11)-(14) capture the "principal-agent" aspect of the problem. Equation (11) states that the optimal process $c_{t}$ cannot be mandated by the government (since the government observes neither the consumption nor the assets of the agent). Instead, the optimal consumption process is chosen optimally by the decision-making agent, who would not impose constraint (9) on her choices, if left alone. Due to the government intervention, the budget dynamics of equation (12) differ from the ones in equation (4) in two ways. First, the modified dynamics reflect the presence of the transfers $d G_{t}$, and second, equation (13) implies that the consumer needs to finance these transfers by paying the amount $D_{0}$ upon entering retirement. Accordingly, an instant after entering retirement, her wealth $W_{0^{+}}$is
equal to the funds she has accumulated in the pre-retirement phase ( $W_{0}$ ) net of the lump sum payment $D_{0}$.

The final requirement that constrains a consumer's choices is the borrowing constraint (14). This constraint plays a central role in the analysis. Without this constraint, it would be impossible for the government to find any set of transfers that would induce the agent to choose a consumption path that satisfies (9). The reason is due to a Ricardian Equivalence: Since the market is dynamically complete in the absence of the constraint (14), a consumer's feasible consumption plans are constrained only by the requirement that the net present value of her consumption be equal to the wealth she has accumulated. Since the net present value of government transfers is equal to the lump sum payment $D_{0}$, the consumer's intertemporal budget constraint is unaffected by the government intervention, no matter what process $G_{t}$ the government chooses. Accordingly, the transfers cannot affect the consumer's plans. Agents can continue to consume as they would in the absence of government intervention and only modify their portfolios so as to undo the effects of the transfers.

The presence of a borrowing constraint such as (14), however, makes transfers nonneutral. The reason is that a borrowing constraint implies stronger restrictions than a simple intertemporal budget constraint on the agent's feasible consumption choices. Hence, by a judicious choice of transfers, the government can affect the agent's consumption. Importantly, the borrowing constraint (14) is realistic and easy to implement in practice. It suffices that the government instruct courts not to enforce agreements that would let lenders seize future government transfers as collateral for loans.

Because of the central role played by the borrowing constraint (14), the next section reviews some known results related to the implications of the constraint (14) for optimal consumption processes. Subsequent sections use these results to solve problem 1.

## 4 The agent's consumption choices in the presence of government intervention and borrowing constraints

Suppose that at the time of retirement (time 0) the government collects an amount $D_{0}$ and then assumes the obligation to deliver an admissible cumulative transfer process $G_{t}$. It is natural to ask how the agent's consumption choices will be affected by this intervention in the presence of the borrowing constraint (14).

To gain some intuition, it is useful to start by assuming that there is no uncertainty ( $\sigma=0$ ), so that $\mu=r$, the stochastic discount factor is deterministic $\left(H_{t}=e^{-r t}\right)$, and the agent's dynamic budget constraint is given by $d W_{t}=(q+r) W_{t} d t-c_{t} d t+d G_{t}$. The deterministic dynamics of $W_{t}, H_{t}$ imply that the constraint $W_{t} \geq 0$ amounts to the requirement ${ }^{19}$

$$
\begin{equation*}
\int_{0}^{t} c_{s} e^{-q s} H_{s} \leq W_{0}-D_{0}+\int_{0}^{t} e^{-q s} H_{s} d G_{s} \text { for all } t \geq 0 \tag{15}
\end{equation*}
$$

Applying the Lagrangian method, an agent's problem can be converted into an unconstrained problem by attaching Lagrange multipliers $\lambda, \zeta_{t} \geq 0$ to obtain

$$
\begin{align*}
\mathcal{L} & =\int_{0}^{\infty} e^{-(\rho+q) t} \frac{c_{t}^{1-\gamma}}{1-\gamma} d t+\lambda\left[W_{0}-D_{0}+\int_{0}^{\infty} e^{-q t} H_{t}\left(d G_{t}-c_{t} d t\right)\right]  \tag{16}\\
& +\left\{\int_{0}^{\infty} \zeta_{t}\left(W_{0}-D_{0}+\int_{0}^{t} e^{-q s} H_{s}\left(d G_{s}-c_{s} d s\right)\right) d t\right\}
\end{align*}
$$

Applying integration by parts to the second line of (16) and imposing the transversality condition $\lim _{t \rightarrow \infty} e^{-q t} H_{t} W_{t}=0$ gives

$$
\begin{equation*}
\mathcal{L}=\int_{0}^{\infty} e^{-(\rho+q) t}\left(\frac{c_{t}^{1-\gamma}}{1-\gamma}-e^{\rho t} \lambda X_{t} H_{t} c_{t}\right) d t+\lambda \int_{0}^{\infty} e^{-q s} H_{s} X_{s} d G_{s}+\lambda\left[W_{0}-D_{0}\right], \tag{17}
\end{equation*}
$$

where $X_{t} \equiv 1-\int_{0}^{t} \frac{\zeta_{s}}{\lambda} d s$. Maximizing $\mathcal{L}$ over $c_{t}$ amounts to simply maximizing the expression

[^10]inside round brackets in equation (17), which gives
\[

$$
\begin{equation*}
c_{t}=\left(\lambda e^{\rho t} H_{t} X_{t}\right)^{-\frac{1}{\gamma}} \tag{18}
\end{equation*}
$$

\]

If all $\zeta_{s}=0$ (i.e., when the borrowing constraint $W_{t} \geq 0$ is not binding) then $X_{t}=1$, and equation (18) amounts to the familiar result that an agent's marginal utility of consumption $\left(e^{-\rho t} c_{t}^{-\gamma}\right)$ be proportional to the stochastic discount factor $H_{t}$.

However, when the borrowing constraint is binding, then consumption is affected by the presence of the decreasing process $X_{t}$, which reflects the cumulative effect of the Lagrange multipliers associated with the borrowing constraint. By construction $X_{t}$ is a process that is non-increasing and starts at $X_{0}=1$.

To fully determine the solution to the consumer's problem, one needs to determine the Lagrange multipliers $\lambda, \zeta_{s}$. He and Pages (1993) show that this amounts to first maximizing $\mathcal{L}$ over $c_{t}$ (given arbitrary $\lambda, X_{t}$ ) and then minimizing the resulting expression over $\lambda, X_{t}$. Specifically, He and Pages (1993) show the following Proposition, which holds also in the presence of uncertainty: ${ }^{20}$

Proposition 1 Let $\mathcal{D}$ be the set of non-increasing, non-negative and progressively measurable processes that start at $X(0)=1$. Then, the value function $V\left(W_{0}\right)$ of an agent can be expressed as:
$V\left(W_{0}\right)=\min _{\lambda>0, X_{s} \in \mathcal{D}}\left[E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \max _{c_{s}}\left(\frac{c_{s}^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s} H_{s} X_{s} c_{s}\right) d s+\lambda \int_{0}^{\infty} e^{-q s} H_{s} X_{s} d G_{s}\right)+\lambda\left(W_{0}-D_{0}\right)\right]$

Let $X_{t}^{*}, \lambda^{*}$ denote the process $X_{t}$ and the constant $\lambda$ that minimize the above expression. Then the optimal consumption process $c_{t}^{*}$ for a consumer faced with the borrowing constraint (14) is given by (18) evaluated at $\lambda=\lambda^{*}, X_{t}=X_{t}^{*}$. Moreover, the process $X_{t}^{*}$ decreases only

[^11]when the associated wealth process $\left(W_{t}\right)$ falls to zero and is otherwise constant, i.e.:
\[

$$
\begin{equation*}
\int_{0}^{\infty} W_{t} d X_{t}^{*}=0 \tag{20}
\end{equation*}
$$

\]

Finally, the resulting wealth process for any $t>0$ is given by

$$
\begin{equation*}
W_{t}=\frac{E_{t}\left(\int_{t}^{\infty} e^{-q(s-t)} X_{s}^{*} H_{s} c_{s}^{*} d s\right)}{X_{t}^{*} H_{t}}-\frac{E_{t}\left(\int_{t}^{\infty} e^{-q(s-t)} X_{s}^{*} H_{s} d G_{s}\right)}{X_{t}^{*} H_{t}} \tag{21}
\end{equation*}
$$

## 5 Government transfers and their welfare effects: an upper bound

Proposition 1 gives an intuitive way to summarize the effects of the incentive compatibility requirement (equations [11]-[14]).

It asserts that every government intervention $\left(G_{t}, D_{0}\right)$ will be associated with a constant $\lambda^{*}\left(G_{t}, D_{0}\right)$ and a Lagrange multiplier process $X_{t}^{*}\left(G_{t}, D_{0}\right)$. Given this correspondence between a choice of $\left(G_{t}, D_{0}\right)$ and the resulting pair $\left(\lambda^{*}, X_{t}^{*}\right)$, there is a straightforward way to obtain an upper bound to the value function of problem 1. In particular consider the following problem:

Problem 2 Maximize:

$$
\begin{equation*}
J\left(W_{0}\right) \equiv \max _{c_{t}, X_{t} \in \mathcal{D}, \lambda>0} E_{0} \int_{0}^{\infty} e^{-(\rho+q) s} \frac{c_{s}^{1-\gamma}}{1-\gamma} d s \tag{22}
\end{equation*}
$$



Figure 1: An illustration of Lemma 1. The admissible choices of problem 1 map into a subset of the admissible choices of problem 2.
subject to:

$$
\begin{align*}
E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s} c_{s} d s\right) & \leq W_{0}  \tag{23}\\
c_{t} & \geq \xi  \tag{24}\\
c_{t} & =\left(\lambda e^{\rho t} H_{t} X_{t}\right)^{-\frac{1}{\gamma}} \tag{25}
\end{align*}
$$

Problem 2 is the problem of a government that can choose directly the consumption of the agent, subject to the intertemporal budget constraint (23), the constraint on the minimum consumption level (equation [24]), and the additional requirement that any chosen consumption process should have a representation in the form of equation (25) for some $X_{t}$. In effect, problem 2 allows the government to choose directly the Lagrange multipliers $\left(\lambda, X_{t}\right)$ without being concerned whether there exists any pair of payments and transfers $\left(G_{t}, D_{0}\right)$ that would render these Lagrange multipliers as shadow values of the consumer's optimization problem (11).

Figure 1 gives a graphical argument to show that the optimized value $J$ of Problem 2 provides an upper bound to the value function of problem 1. Clearly, any admissible pair $G_{t}, D_{0}$ needs to induce a consumption process that satisfies (24). Additionally, because
transfers are fully funded, they don't alter the consumer's intertemporal budget constraint, and hence any admissible consumption process of problem 1 needs to satisfy ${ }^{21}$ equation (23). Moreover, Proposition 1 asserts that there always exists some pair of $\lambda, X_{t}$ such that any admissible consumption process of problem 1 can be expressed in the form of equation (25). Therefore, any admissible $G_{t}, D_{0}$ maps into a subset of pairs $\left(X_{t}, \lambda\right)$ allowed by Problem 2, and the value function of problem 2 must therefore provide an upper bound to problem 1. The following Lemma provides a formal statement.

Lemma 1 Let $\mathcal{G}$ be the class of all transfer processes $G_{t}$ that enforce (9) and satisfy (10). Furthermore, let $V\left(W_{0}\right)$ be given as in equation (19). Then the value functions of problems 1 and 2 are related by

$$
\begin{equation*}
\Omega\left(W_{0}\right)=\max _{D_{0}, G_{t} \in \mathcal{G}} V\left(W_{0}\right) \leq J\left(W_{0}\right) \tag{26}
\end{equation*}
$$

The remainder of this section derives an explicit solution to problem 2, while the next section shows that there exist transfer processes $G_{t}^{*}$ that are optimal, because they make (26) hold with equality.

As a first step towards solving problem 2, it is useful to ask whether constraints (23), (24), and (25) bind at an optimum. The top panel of figure 2 gives an optimal consumption path for a random realization of $H_{t}$ assuming that one maximizes (22) subject only to the intertemporal budget constraint (23). The resulting solution is $c_{t}^{* * *}=\left(\lambda^{* * *} e^{\rho t} H_{t}\right)^{-\frac{1}{\gamma}}$ and it corresponds to what the consumer would choose, if left alone. Because $H_{t}$ is log-normal, so is $c_{t}$ and accordingly $c_{t}<\xi$ with positive probability. Imposing the constraint $c_{t} \geq \xi$ (but not the constraint [25]) leads to the optimal consumption path $c_{t}^{* *}=\max \left[\xi,\left(\lambda^{* *} e^{\rho t} H_{t}\right)^{-\frac{1}{\gamma}}\right] .{ }^{22}$
${ }^{21}$ The consumer's dynamic budget constraint (12) implies the intertemporal budget constraint

$$
W_{0}-D_{0}+\int_{0}^{\infty} e^{-q t} H_{t}\left(d G_{t}-c_{t} d t\right) \geq 0
$$

Combining the intertemporal budget constraint with condition (10) implies (23).
${ }^{22}$ Clearly, $\left(\lambda^{* *}\right)^{-\frac{1}{\gamma}}<\left(\lambda^{* * *}\right)^{-\frac{1}{\gamma}}$, otherwise it would be impossible that both $c_{t}^{* * *}$ and $c_{t}^{* *}$ satisfy (23).


Figure 2: Implications of the constraints in problem 2

The solution $c_{t}^{* *}$ is what the government would choose, if it could directly observe and mandate the agent's consumption and portfolio choices.

However, the government cannot directly observe these choices. Instead, it needs to induce the agent to choose consumption paths that satisfy $c_{t} \geq \xi$, by exploiting binding borrowing constraints. This is captured by equation (25). The bottom panel of Figure 2 shows that this incentive compatibility requirement is in general binding. Indeed, equation (25) implies that any admissible consumption process should satisfy the property that the ratio $c_{t} / c_{t}^{* * *}=\left(\frac{\lambda}{\lambda^{* * *}}\right)^{-\frac{1}{\gamma}} X_{t}^{-\frac{1}{\gamma}}$ should be a non-decreasing process (since $X_{t}$ is non-increasing). Clearly, the ratio $c_{t}^{* *} / c_{t}^{* * *}$ has decreasing sections and therefore $c_{t}^{* *}$ cannot satisfy (25). Therefore, $J\left(W_{0}\right)$ (and accordingly the value function $\Omega\left(W_{0}\right)$ in problem 1 ) will in general be lower
than what the government could attain if it observed and mandated consumption.
The next proposition determines the solution of problem 2:

Proposition 2 Let the constants $\phi, K$ be defined $a s^{23}$

$$
\begin{align*}
\phi & \equiv \frac{-\left(\rho-r-\frac{\kappa^{2}}{2}\right)+\sqrt{\left(\rho-r-\frac{\kappa^{2}}{2}\right)^{2}+2(\rho+q) \kappa^{2}}}{\kappa^{2}}>1,  \tag{27}\\
K & \equiv \frac{\gamma}{\frac{\gamma-1}{\gamma} \frac{\kappa^{2}}{2}+\gamma(r+q)+(\rho-r)}, \tag{28}
\end{align*}
$$

and assume that

$$
\begin{equation*}
W_{0} \geq W^{\min } \equiv \frac{\frac{1}{\gamma}+\phi-1}{\phi-1} K \xi \tag{29}
\end{equation*}
$$

Additionally, for any $\lambda>0$, let the process $X_{t}^{*}$ be given by

$$
\begin{equation*}
X_{t}^{*}(\lambda) \equiv \min \left[1, \frac{\xi^{-\gamma} / \lambda}{\max _{0 \leq s \leq t}\left(e^{\rho s} H_{s}\right)}\right] \tag{30}
\end{equation*}
$$

Then the value function of problem (2) is given by
$J\left(W_{0}\right)=\min _{\lambda \geq 0}\left[E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{1-\frac{1}{\gamma}}}{1-\gamma} d s-\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s+\lambda W_{0}\right)\right]$

$$
\begin{equation*}
=\min _{\lambda \geq 0}\left[-\frac{K \xi^{1-\gamma}}{\gamma \phi(\phi-1)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi}+K \frac{\gamma}{1-\gamma} \lambda^{1-\frac{1}{\gamma}}+\lambda W_{0}\right] . \tag{32}
\end{equation*}
$$

Letting $\lambda^{*}$ be the scalar that minimizes (32), the optimal triplet that solves problem (2) is
${ }^{23}$ To see why $\phi>1$, notice that $\phi$ solves the quadratic equation

$$
\frac{\kappa^{2}}{2} \phi^{2}+\left(\rho-r-\frac{\kappa^{2}}{2}\right) \phi-(\rho+q)=0
$$

Evaluating the left hand side of this equation at $\phi=1$ gives $-(r+q)<0$. Hence the larger of the two roots of the quadratic equation is larger than 1.
given by $\lambda^{*}, X_{t}^{*}=X_{t}\left(\lambda^{*}\right)$, and $c_{t}^{*}=\left(\lambda^{*} e^{\rho t} H_{t} X_{t}^{*}\right)^{-\frac{1}{\gamma}}$.

Proposition 2 provides an explicit expression for the value function of problem 2, assuming that the agent enters retirement with a level of assets that are no smaller than the lower bound of equation (29). For now, equation (29) will be assumed to be satisfied. Section 7 derives an optimal process of pre-retirement savings ensuring that (29) holds.

## 6 Optimal Transfer Processes

This section illustrates two optimal distinct processes $G_{t}^{*}$ that attain the upper bound $V\left(W_{0} ; G_{t}^{*}\right)=J\left(W_{0}\right)$.

### 6.1 A constant income stream

The simplest form of government transfer process is a constant income stream: The government collects a lump sum tax of $D_{0}=\frac{y_{0}}{r+q}$ and in exchange it delivers a constant stream of $y_{0}$ until the agent dies. Surprisingly, this simple policy is optimal, as long as $y_{0}$ is chosen judiciously. The following proposition gives a closed form solution for $y_{0}$.

Proposition 3 Let $y_{0}$ be given by

$$
\begin{equation*}
y_{0} \equiv(r+q) K \xi\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right) \tag{33}
\end{equation*}
$$

where $K$ is given in (28) and $\phi$ is given in (27). The policy of collecting $D_{0}=\frac{y_{0}}{r+q}$ and providing transfers equal to $y_{0}$ until the agent dies, attains the upper bound $V\left(W_{0} ; G_{t}=y_{0}\right)=$ $J\left(W_{0}\right)$ and is therefore optimal.

An interesting feature of the optimal policy in proposition 3 is contained in the following Lemma

Lemma 2 The optimal policy of proposition 3 has the property

$$
\frac{y_{0}}{\xi}>1
$$

Lemma 2 shows that if the government wants to ensure a minimum consumption of one dollar, it needs to deliver more than one dollar in guaranteed income. This result is driven by the fact that agents cannot be excluded from markets. To see why, suppose that -contrary to Lemma 2- the government were to set $y_{0}=\xi$, and consider a retiree with current wealth $W_{t}=0$. Since the borrowing constraint is binding for that retiree, her Euler equation implies that her current marginal utility of consumption will be no smaller than the expected value of marginal utility tomorrow, discounted by the subjective discount rate and compounded by the interest rate. In general, this implies that the retiree will find it optimal to make a savings and portfolio choice today, so that tomorrow's consumption will exceed today's consumption with positive probability. But, the dynamic budget constraint implies that the only way to achieve such an optimal outcome is to set $c_{t}<y_{0}=\xi$ today, which would violate the requirement $c_{t} \geq \xi$. Alternatively phrased, since in general a retiree will find it optimal to set $c_{t}<y_{0}$ every time the borrowing constraint is binding, the optimal value of $y_{0}$ needs to exceed $\xi$.

### 6.2 Portfolio Insurance

Providing agents with a constant income is not the unique optimal way to attain the upper bound in Proposition 2. Moreover, the optimality of a constant income stream depends crucially on the assumption of a constant investment opportunity set (constant interest rate and market price of risk). In this section we present a generic approach to constructing income processes that attain the upper bound of Proposition 2. An advantage of the approach presented here is that it does not depend on any assumptions about the stochastic discount factor. In particular the form of the optimal process for $G_{t}$ would remain optimal, when the
model is closed in general equilibrium. (Section 8 provides further discussion.)
To describe this approach, let $\lambda^{*}$ be the scalar that minimizes (32). Then define the government's transfer process as:

$$
\begin{equation*}
d G_{t}=-\left(\frac{1}{\gamma}+\phi-1\right) K \xi \frac{d X_{t}^{*}}{X_{t}^{*}} \tag{34}
\end{equation*}
$$

where $X_{t}^{*}\left(\lambda^{*}\right)$ is the process defined in (30).
This section shows the following two results: First, the process (34) attains the upper bound of Proposition 2. Second, the process (34) has an intuitive economic interpretation as a type of minimum return guarantee (portfolio insurance) on the agent's optimal portfolio of stocks and bonds.

The following proposition formalizes the first claim and provides results that are useful towards establishing the second claim.

Proposition 4 Let $\lambda^{*}$ be the scalar that minimizes (32) and $X_{t}^{*}\left(\lambda^{*}\right)$ be the process that is given in (30). Consider an agent who anticipates transfers given by (34) and is faced with an initial tax of $D_{0}$, where $D_{0}$ satisfies (10). Then
a) her value function coincides with the upper bound given in (32).
b) Letting

$$
\begin{equation*}
Z_{t} \equiv \lambda^{*} e^{\rho t} H_{t} X_{t}^{*} \tag{35}
\end{equation*}
$$

the agent invests

$$
\begin{equation*}
\pi_{t}=\frac{\kappa}{\sigma} K \xi\left[(\phi-1)\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi-1}+\frac{1}{\gamma}\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{-\frac{1}{\gamma}}\right] \tag{36}
\end{equation*}
$$

dollars in the stock market and consumes

$$
\begin{equation*}
c_{t}=Z_{t}^{-\frac{1}{\gamma}} \tag{37}
\end{equation*}
$$

while the agent's optimal wealth process $W_{t}$ is given by

$$
\begin{equation*}
W_{t}=-K \xi\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi-1}+K Z_{t}^{-\frac{1}{\gamma}} \tag{38}
\end{equation*}
$$

c) The initial payment $D_{0}$ associated with (34) is given by

$$
\begin{equation*}
D_{0}=K \xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\left(\frac{\lambda^{*}}{\xi^{-\gamma}}\right)^{\phi-1} . \tag{39}
\end{equation*}
$$

The portfolio policy (36) will aid in the interpretation of (34) as a form of portfolio insurance. To obtain some intuition on the nature of (34), consider first the following puzzling feature of the optimal portfolio policy: As $c_{t} \rightarrow \xi$, equation (37) implies that $Z_{t} \rightarrow \xi^{-\gamma}$ and (38) implies that $W_{t} \rightarrow 0$. However, the agent's holdings of stock satisfy

$$
\begin{equation*}
\lim _{Z_{t} \rightarrow \xi^{-\gamma}} \pi_{t}=\left(\frac{1}{\gamma}+\phi-1\right) K \xi \frac{\kappa}{\sigma}>0 \tag{40}
\end{equation*}
$$

Because the agent's financial wealth approaches zero as $Z_{t} \rightarrow \xi^{-\gamma}$, but her stock position doesn't, a further negative return on the stock market would lead to a negative financial asset position in the absence of any transfers. To prevent such a negative asset position, the transfers given by (34) act as a minimum return guarantee, which ensures that the agent receives just enough funds to sustain her financial wealth at zero and keep her consumption at $\xi$.

It is useful here to clarify that these transfers do not require that the government actually observe the path of the agent's assets or her consumption. By the definition of $X_{t}^{*}$ in equation (30), the government only needs to know the evolution of the stochastic discount factor $H_{t}$,
which can be inferred from the path of the stock market ${ }^{24,25}$.
A simple way of thinking about the transfer process $G_{t}$ in (34) is that the government and the agent have a joint understanding of how the consumer will consume and invest in the presence of the transfers given by (34). Based on its (correct) understanding of the consumer's optimal policies, the government can infer the agent's wealth and make just enough transfers when needed, so as to keep the agent's wealth above 0 and her optimal consumption above $\xi$.

### 6.3 Comparing the two policies

Given that both policies attain the upper bound of equation (32), this means that they are equivalent from a welfare perspective. The derivations in the appendix also show that they imply exactly the same consumption process "path by path".

However, the two policies do differ. They make transfers of different magnitudes in different states of the world. The initial payments that they imply are also different. Indeed, the initial payment associated with the constant income policy is:

$$
\begin{equation*}
D_{0}^{\text {const. }}=\frac{y_{0}}{r+q}=K \xi\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right), \tag{41}
\end{equation*}
$$

whereas by equation (39), the initial payment of the portfolio insurance policy is:

$$
\begin{equation*}
D_{0}^{p . i .}=K \xi\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right)\left(\frac{\lambda^{*}}{\xi^{-\gamma}}\right)^{\phi-1} \tag{42}
\end{equation*}
$$

[^12]Since $c_{0} \geq \xi \operatorname{and}^{26} c_{0}^{-\gamma}=\lambda^{*}$, it follows that $\lambda^{*} / \xi^{-\gamma} \leq 1$ and accordingly $\frac{D_{0}^{p . i .}}{D_{0}^{\text {onst. }}} \leq 1$. Hence the "portfolio insurance" policy implies an initial payment that cannot be larger than the initial payment of the "constant income" policy. This is intuitive, since the constant income policy delivers the same transfers in all states of the world, including states of the world where the borrowing constraint doesn't bind. By contrast, the "portfolio insurance" policy delivers payments only when the borrowing constraint binds.

However, when $c_{0}=\xi$ (or alternatively $W_{0^{+}}=0$ ) the two policies imply the same initial payment. Hence, the initial payment of the two policies differs only when the borrowing constraint is not binding, but is identical when the borrowing constraint does bind. This is the reason why the two policies imply different initial payments, but are identical from a welfare perspective. The additional resources delivered by the constant income policy are delivered in states of the world where the borrowing constraint is not binding and hence can be "undone" by agents' portfolio choice.

The above discussion illustrates that simply comparing the costs of alternative retirement benefit guarantees does not provide sufficient information for welfare comparisons.

## 7 Minimum level of assets and implications for preretirement savings

A maintained assumption of the analysis sofar was that the agent's assets upon entering retirement were above the minimum level of equation (29). As the next Proposition shows, this assumption is not only sufficient, but it is also necessary for the existence of any transfer processes that can induce a consumption process that satisfies $c_{t} \geq \xi$.

Proposition 5 An admissible transfer process $G_{t}$ that can induce $c_{t} \geq \xi$ post-retirement exists if and only if (29) holds, i.e. if $W_{0} \geq W^{\min }$.

[^13]Proposition 5 has implications for the government's pre-retirement problem. Specifically, the feasibility of enforcing the constraint $c_{t} \geq \xi$ post-retirement is equivalent to requiring that the agent arrive in retirement with assets that are at least as large as $W^{\text {min }}$.

Therefore, prior to retirement, the government needs to ensure that the agent saves an adequate fraction of labor income, so as to be able to finance the post-retirement optimal transfer processes $G_{t}$, which were described previously. Specifically, recalling that $t^{b}$ is an agent's date of birth, $\tau$ the duration of work, and $t^{b}+\tau$ is the time of retirement (which is normalized to zero), the government can collect pre-retirement payments from the agent equal to $\int_{t^{b}}^{0} d S_{t}, d S_{t} \geq 0$, and rebate a lump-sum amount $L_{0}$ at retirement so that

$$
\begin{equation*}
W_{0} \equiv W_{0^{-}}+L_{0} \geq W^{\min } \tag{43}
\end{equation*}
$$

and ${ }^{27}$

$$
\begin{equation*}
E_{t^{b}} e^{-q \tau}\left(\frac{H_{0}}{H_{t^{b}}}\right) L_{0}=E_{t^{b}} \int_{t^{b}}^{0} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) d S_{t} . \tag{44}
\end{equation*}
$$

Equation (43) requires that the agent's assets upon entering retirement ( $W_{0}$ ) - which comprise the agent's assets an instant before retirement ( $W_{0^{-}}$) and the governmental lumpsum transfer $\left(L_{0}\right)$ - be at least as large as $W^{\text {min }}$. Additionally, equation (44) is analogous to equation (10), since it requires that the present value of the lump sum transfer $L_{0}$ be equal to the present value of the pre-retirement payments $d S_{t}$. Because of equation (44), I refer to the transfers $d S_{t}$ as "mandatory savings" rather than "distortionary labor taxes", since they are returned to the agent (compounded at a fair market return) in the form of a lump sum transfer at retirement. (A practical implication of this difference would arise in an extended model with endogenous, continuous labor supply, since taxes would distort the intra-temporal first order conditions for labor supply, whereas mandatory savings would

[^14]not.)
An appealing feature of the post-retirement government policies of the previous sections is that they did not actually require the government to implement and administer them. The government could simply mandate that retirees purchase an "insurance" policy with payoffs $d G_{t}$, and then private competitive entities (insurance companies, pension funds, etc.) could provide this policy in exchange for an upfront fee $D_{0}$.

To ensure that the same "decentralization" through private entities is also feasible for the pre-retirement problem, I make an additional assumption on the allowable combinations of $S_{t}, L_{0}$. To motivate this assumption, suppose that the government determined a policy pair $\left(S_{t}, L_{0}\right)$, and mandated that working agents make transfers to a pension fund equal to $d S_{t}$, in exchange for a transfer payment of $L_{0}$ from the pension fund at retirement. Then, the financial assets of the pension fund are given by $\widetilde{W}_{t}$, with $\widetilde{W}_{t}$ defined as

$$
\begin{equation*}
\widetilde{W}_{t} \equiv E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) L_{0}-E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) d S_{u} \tag{45}
\end{equation*}
$$

Equation (45) follows from the fact that the financial assets of the pension fund ( $\widetilde{W}_{t}$ ) plus the remaining present value of transfers from the workers $E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) d S_{u}$ must be equal to the present value of the lump sum transfer to be paid once the agents retire $E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) L_{0}$. To prevent default in the spirit of Bulow and Rogoff (1989), ${ }^{28}$ I require that

$$
\begin{equation*}
\widetilde{W}_{t} \geq 0 \Longleftrightarrow E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) L_{0}-E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) d S_{u} \geq 0 \tag{46}
\end{equation*}
$$

[^15]for all $t \in\left[t^{b}, 0\right]$. This non-negativity of financial assets for a pension fund is the direct analog of the non-negativity of financial assets for consumers (equation [6]). ${ }^{29}$ It is noteworthy that the requirement (46) is automatically satisfied for any deterministic policy pair $S_{t}, L_{0}$ that satisfies (44).

Finally, to ensure that there exist some feasible combination of $S_{t}, L_{0}$ that satisfy (44), (46), and can enforce (43), I assume that the present value of $W^{\text {min }}$ as of the birth of the agent is no larger than the respective present value of the agent's income

$$
\begin{equation*}
E_{t^{b}} e^{-q \tau}\left(\frac{H_{0}}{H_{t^{b}}}\right) W^{\min }<E_{t^{b}} \int_{t^{b}}^{0} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) Y d t \tag{47}
\end{equation*}
$$

Using $E_{t^{b}}\left(\frac{H_{t}}{H_{t^{b}}}\right)=e^{-r\left(t-t^{b}\right)}$ inside (47), taking logarithms on both sides and rearranging leads to the following re-statement of (47) in terms of an agent's minimum duration of work, which is maintained throughout

$$
\begin{equation*}
\tau>\frac{\log \left(1+(r+q) \frac{W^{\min }}{Y}\right)}{(r+q)} \tag{48}
\end{equation*}
$$

The government's pre-retirement problem can now be summarized in a manner analogous to the government's post-retirement problem 1.

Problem 3 Choose $S_{t}, L_{0}$ so as to maximize

$$
\begin{equation*}
\Omega_{t^{b}} \equiv \max _{S_{t}, L_{0}} E_{t^{b}} \int_{t^{b}}^{0} e^{-(\rho+q)\left(t-t^{b}\right)} \frac{\left(c_{t}\right)^{1-\gamma}}{1-\gamma} d t+E_{t^{b}} e^{-(\rho+q) \tau} J\left(W_{0^{-}}+L_{\tau}\right) \tag{49}
\end{equation*}
$$

subject to (43), (44), (46), and subject to the constraint that $W_{t}$ is the wealth process that results for the choices of $c_{t}, \pi_{t}$ that solve the decision-making agent's pre-retirement opti-

[^16]mization problem given $S_{t}, L_{\tau}$ :
\[

$$
\begin{equation*}
\max _{\left\langle c_{t}, \pi_{t}\right\rangle} E_{t^{b}} \int_{t^{b}}^{0} e^{-(\rho+q)\left(t-t^{b}\right)} \frac{\left(c_{t}\right)^{1-\gamma}}{1-\gamma} d t+E_{t^{b}} e^{-(\rho+q) \tau} J\left(W_{0^{-}}+L_{\tau}\right) \tag{50}
\end{equation*}
$$

\]

s.t.:

$$
\begin{align*}
d W_{t} & =q W_{t} d t+\pi_{t}\left\{\mu d t+\sigma d B_{t}\right\}+\left\{W_{t}-\pi_{t}\right\} r d t+\left(Y-c_{t}\right) d t-d S_{t}  \tag{51}\\
W_{t} & \geq 0 \text { for all } t \geq t^{b} . \tag{52}
\end{align*}
$$

Equation (49) requires that the government choose mandatory saving policies $S_{t}$ and a lump sum transfer upon retirement so as to maximize the agent's life-time expected utility at birth. The equations (50) - (52) require that the wealth process is the result of the agent's optimal consumption and portfolio choices $c_{t}, \pi_{t}$ that result in the presence of the government policies $S_{t}, L_{0}$.

The discrepancy in the government's and the agent's objectives in the pre-retirement problem stems from the fact that the government wants to ensure that the agent arrives in retirement with a minimum amount of assets (in order satisfy the constraint $c_{t} \geq \xi$ postretirement). If left alone, the agent would not necessarily arrive in retirement with such a minimum level of assets. Accordingly, the government policies $S_{t}, L_{0}$ need to induce her to choose pre-retirement consumption and portfolio policies that will result in a minimum level of retirement assets.

As a first step towards solving problem 3, it is useful to consider the solution of the following problem

Problem 4 Choose $c_{t}, W_{0}$ so as to maximize

$$
J_{t^{b}} \equiv \max _{c_{t}, \pi_{t}} E_{t^{b}} \int_{t^{b}}^{0} e^{-(\rho+q)\left(t-t^{b}\right)} \frac{\left(c_{t}\right)^{1-\gamma}}{1-\gamma} d t+E_{t^{b}} e^{-(\rho+q) \tau} J\left(W_{0}\right)
$$

subject to the dynamic budget constraint (5), the non-negativity of wealth constraint (6) and
the additional constraint $W_{0} \geq W^{\mathrm{min}}$.

Problem 4 is the problem that would be solved by the agent, if she voluntarily imposed the constraint $W_{0} \geq W^{\text {min }}$ on her own choices. As one might expect, the fact that the agent voluntarily imposes the constraint $W_{0} \geq W^{\min }$ on her own decisions implies that the value function of problem 4 is an upper bound to the solution of problem 3. This is formalized in the next Lemma

Lemma 3 For $\Omega_{t^{b}}, J_{t^{b}}$ denoting the value functions of problems 3 and 4 respectively, $\Omega_{t^{b}} \leq$ $J_{t^{b}}$.

Variants of problem 4 have been studied elsewhere, and especially in the literature on portfolio insurance. (See e.g., Basak (2002)). The new aspect of this paper is that the solution of Problem 4 acts as an upper bound to Problem 3 and hence can be used to check the optimality of various mandatory savings programs.

Solving Problem 4 in closed form is difficult, because of the presence of the borrowing constraint $W_{t} \geq 0$ and the extra state variable introduced by the agent's distance to retirement. Fortunately, the exact solution of problem 4 is not required for the analysis that follows. Instead, it is sufficient to establish the following property of any optimal solution to problem 4.

Lemma 4 If $W_{t}^{*}$ is the optimal wealth process associated with problem 4, then there exist a time $\chi=-\frac{\log \left(1+(r+q) \frac{W^{\text {min }}}{Y}\right)}{(r+q)} \in\left(t^{b}, 0\right)$ such that $W_{t}^{*}>0$ for all $t \in(\chi, 0]$.

Lemma 4 asserts that there exists a time $\chi$ prior to retirement, such that the borrowing constraint $W_{t}^{*} \geq 0$ is non-binding for any $t \in[\chi, 0)$. The next proposition shows that a simple way to construct an optimal policy to the government's problem 3 is to set $d S_{t}>0$ only during $[\chi, 0)$.

## Proposition 6 Consider the government policy given by

$$
d S_{t}=\left\{\begin{array}{c}
0 \text { if } t \in\left[t^{b}, \chi\right) \\
Y d t \text { if } t \in[\chi, 0]
\end{array}\right.
$$

and $L_{0}=W^{\mathrm{min}}$. Then this policy is optimal, since the associated value function $V_{t^{b}}$ of the agent satisfies $V_{t^{b}}=J_{t^{b}}$.

Simply put, proposition 6 asserts that an optimal solution to the government's problem is to make a transfer equal to $W^{\text {min }}$ at retirement and finance this transfer by requiring mandatory savings only for a few years immediately prior to retirement. This "backloading" of savings is driven by Lemma 4 and in particular, the observation that the borrowing constraint stops binding for some time prior to retirement $(t \in[\chi, 0])$.

Intuitively, by the time $\chi$, the agent has accumulated enough wealth that income can be channeled towards savings without distorting the optimal consumption decisions the agent would have made, if she was solving problem 4.

Summarizing, this section has shown that the feasibility of any fully funded post-retirement plan $d G_{t}$ that can provide a minimum standard of living in retirement, is equivalent to the requirement that the agent arrive in retirement with assets at least as large as $W^{\mathrm{min}}$. In order to ensure that the agent has accumulated this amount of assets by retirement, the government needs to enforce a minimum amount of savings pre-retirement. Given the frictions implied by the presence of borrowing constraints, the current framework suggests that these mandatory savings should be done just prior to retirement, when the agent has accumulated enough wealth to avoid consumption distortions.

Of course, Proposition 6 is mostly of theoretical interest: It helps highlight the fact that borrowing constraints make it optimal to postpone mandatory savings to the years prior to retirement. The stark difference in optimal mandatory savings before $\chi$ and after $\chi$ is sensitive to the maintained assumption of an exogenous retirement time. It is reasonable to
conjecture that endogenizing the retirement time along the lines of Farhi and Panageas (2007) would lead to a smoother age-dependent mandatory savings profile. This extension would be lengthy and would require a separate paper. However, it is likely that the presence of borrowing constraints would still tend to backload mandatory savings, albeit not completely.

## 8 Arbitrary stochastic discount factors and general equilibrium

The assumption of a small open economy facilitated the analysis by rendering the stochastic discount factor exogenous to the model. Another simplifying assumption is that everything is driven by a single shock. Neither of these assumptions is restrictive. Even if the stochastic discount factor were endogenous and driven by multiple sources of uncertainty, most of the results of the paper would survive.

Specifically, the fact that equation (31) provides an upper bound to problem 1 remains valid for any continuous stochastic discount factor $H_{t}$. It is also straightforward to show that an appropriately re-parametrized version the portfolio insurance policy would attain the upper bound of proposition 2 for any continuous stochastic discount factor, and accordingly for the general equilibrium version of the present model. However, the result that depends crucially on the functional form of the stochastic discount factor (3) is the optimality of the constant income policy.

In summary, the qualitative findings of the model would survive even in a closed, general equilibrium economy. ${ }^{30}$ Even though the stochastic discount factor and the price of all guarantees would change, most of the key results of the paper, namely the nature of the upper bound of equation (31), and the optimality of the portfolio insurance policy would

[^17]remain unchanged.

## 9 Conclusion

By exploiting borrowing restrictions of agents, this paper proposed a framework to discuss optimal transfer processes that can ensure a minimum standard of living in retirement.

Within the framework of a baseline life-cycle model, featuring a constant interest rate and market price of risk, two policies were shown to be optimal: According to the first policy, retirees use part of their accumulated assets to purchase a fixed annuity that pays off a constant income stream. The second policy is an appropriate form of portfolio insurance that ensures retirees against further negative returns, once their assets approach zero. Optimal transfers are financed by mandating pre-retirement savings, which optimally take place in the years leading up to retirement, and not at the beginning of the life-cycle.

In summary, the baseline framework supports a mandate for agents to purchase a fixed annuity financed by compulsory savings in the years prior to retirement. However, the optimality of a mandatory annuity depends crucially on interest rates and the market price of risk being constant. This fact lends some support to the critics of mandatory fixed annuities, who point out that such annuities may be suboptimal in a world of historically low real interest rates. Alternative policies, such as the portfolio insurance policies constructed in this paper remain optimal for arbitrary assumptions on the investment opportunity set. However, they are more complex, which may make them unattractive from a practical perspective.

Several issues are unexplored by the present paper. A first issue concerns unobserved preference heterogeneity. If agents have different risk aversions, or subjective discount factors, then the government needs to offer menus of contracts in the spirit of discriminatory pricing. An open question is whether the need to enforce sorting into different types of contracts would affect the qualitative features of the guarantees. A further extension of the present model would be to allow agents to choose their retirement time endogenously and
examine the implications for pre-retirement savings. Studying these two questions is left for future research.

## Appendix

## A Justification for a minimum standard of living $c_{t} \geq \xi$

From a rational perspective, one potential reason for safeguarding that agents can selffinance a minimum standard of living in retirement is to deter them from over-burdening the (distortionary-tax financed) welfare system by claiming welfare benefits, when they are not the intended recipients of such benefits. In that sense agents' financial decisions can be a source of systemic risk, since they can constitute a negative externality for the economic system.

To substantiate this claim, I enrich the model and introduce a simple reason for the existence of a welfare system, along with a stylized model of the welfare system. Specifically, assume that up until retirement agents are identical in every respect. Upon entering retirement, however, a small fraction $\theta$ of agents experiences an unobservable and idiosyncratic shock that results in a random, negative and bounded income stream of $-\overline{Y_{t}}$ for the rest of their lives. (This assumption can be relaxed; individuals could experience the shock at some privately observed, random time after retirement without affecting the intuitions and conclusions). The remaining $1-\theta$ fraction of the agents remain identical to the agents described in the paper sofar. The idiosyncratic shock is catastrophic, in the sense that no agent could self-insure against that shock by accumulating savings

$$
\begin{equation*}
\int_{t^{b}}^{0} e^{-(q+r)\left(t-t^{b}\right)} Y d t<E_{t_{b}} \int_{0}^{\infty} e^{-(q+r)\left(t-t^{b}\right)} \overline{Y_{t}} d t . \tag{53}
\end{equation*}
$$

Equation (53) states that even if an agent saved all her labor income, the resulting present value would still be smaller than the present value of the negative shock $\overline{Y_{t}}$. Accordingly, if self-insurance were the only available form of insurance, then for any $\theta>0$ agents would be faced with a positive probability of an unboundedly negative post-retirement value function.

Because of these catastrophic and unobservable idiosyncratic shocks, the government can raise the welfare of the time $t^{b}$-cohort of agents by creating a "welfare" system, which aims to ensure agents against catastrophic idiosyncratic shocks. The fact that all individual characteristics (shocks, consumption, wealth, $\overline{Y_{t}}$ etc.) are unobservable, and the government can only provide transfers based on public information without being able to affect agents' access to markets, requires some way to ensure that agents truthfully declare whether they suffered a shock. In reality, non-pecuniary costs (such as standing in queues, filing paper-work
etc.) can help "separate" different types of agents. To formalize this notion, suppose that the welfare system works as follows: any agent who enters retirement can obtain transfers $\left(d N_{t} \geq 0\right)$ from the government, by incurring non-pecuniary, time-related $\operatorname{costs} \xi^{-\gamma} d N_{t}$, such as standing in a queue. ${ }^{31}$ (An alternative interpretation is that the agent gets hired and paid a wage $d N_{t}$ for performing some task of low - for illustration purposes say zero- market value, but with a time-related disutility to herself given by $\xi^{-\gamma} d N_{t} .{ }^{32}$ ) Accordingly, the agent's objective is to maximize

$$
\begin{equation*}
E_{0}\left(\int_{0}^{\infty} e^{-(\rho+q) t} \frac{c_{t}^{1-\gamma}}{1-\gamma} d t-\int_{0}^{\infty} e^{-(\rho+q) t} \xi^{-\gamma} d N_{t}\right) \tag{54}
\end{equation*}
$$

The assumption of a constant cost of time $\left(\xi^{-\gamma}\right)$ per unit of transfer simplifies the analysis, but is not key. ${ }^{33}$ As will become clear shortly, the crucial feature of (54) is that the nonpecuniary costs act as a useful screening device so as to separate the agents who have experienced idiosyncratic shocks from those who haven't.

A final assumption is that the transfers $d N_{t}$ are financed by distortionary labor taxes when the cohort of agents is still working. Specifically, the government collects a labor tax equal to $\bar{\omega} Y$, during the work years of the agents, so as to finance any welfare transfers $d N_{t}$ later on. An important difference between the mandatory savings discussed in section 7 and the labor $\operatorname{tax} \bar{\omega} Y$ is that this tax is withheld by the government and redistributed only to agents who experience idiosyncratic shocks, so that there is no direct linkage between the taxes paid and the benefits received by an individual. As is well understood in the literature (see e.g., Barro (1979), Lucas and Stokey (1983)), this decoupling leads to a distortion of the labor-leisure tradeoff and results in deadweight costs. Even though it is straightforward to model such distortions explicitly (see, e.g., Panageas (2010)) for the purposes of this paper it suffices to treat such deadweight costs in a simple reduced-form way and assume that only a fraction $(1-\delta) \bar{\omega} Y$ of an agent's taxes reaches the government. (The literature sometimes refers to such a simple modeling of deadweight costs as "iceberg" costs). The constant $\delta$ captures the fraction of income that gets "wasted" due to work-disincentives. The resulting

[^18]budget constraint of the government is given by
$$
\int_{t^{b}}^{0} e^{-(\rho+q)\left(t-t^{b}\right)}(1-\delta) \bar{\omega} Y=\theta E_{t_{b}} \int_{0}^{\infty} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) d N_{t}
$$
assuming that only the $\theta$-fraction of agents who suffer the idiosyncratic shock ever request transfers.

Letting $i=\mathcal{I}$ if agent $i$ has experienced an idiosyncratic shock and $i=\mathcal{N I}$ otherwise, the "truth-telling" requirement ${ }^{34}$ can be expressed as

$$
\begin{equation*}
\int_{0}^{\infty} d N_{t}^{i}=0, \text { whenever } i=\mathcal{N I} \tag{55}
\end{equation*}
$$

Using the shorthand notation $\theta^{\mathcal{I}} \equiv \theta, \theta^{\mathcal{N I}} \equiv 1-\theta$ and $\tilde{Y}_{t}^{\mathcal{I}} \equiv \bar{Y}_{t}, \tilde{Y}_{t}^{\mathcal{N I}} \equiv 0$, the full government's post-retirement problem can be expressed as

Problem 5 Choose $G_{t}, D_{0}$ to maximize

$$
\begin{equation*}
\max _{G_{t}, D_{0}} E_{0} \sum_{i=\mathcal{I}, \mathcal{N I}} \theta^{i}\left(\int_{0}^{\infty} e^{-(\rho+q) t} \frac{\left(c_{t}^{i}\right)^{1-\gamma}}{1-\gamma} d t-\int_{0}^{\infty} e^{-(\rho+q) t} \xi^{-\gamma} d N_{t}^{i}\right) \tag{56}
\end{equation*}
$$

subject to (10), (55), and subject to the constraint that $c_{t}^{i}, d N_{t}^{i}$ solve the agent's optimization problem given $G_{t}$

$$
\begin{equation*}
c_{t}^{i}=\arg \max _{\left\langle c_{t}^{i}, \pi_{t}^{i}, d N_{t}^{i}\right\rangle} E_{0} \int_{0}^{\infty} e^{-(\rho+q) t} \frac{\left(c_{t}^{i}\right)^{1-\gamma}}{1-\gamma} d t-\int_{0}^{\infty} e^{-(\rho+q) t} \xi^{-\gamma} d N_{t}^{i} \tag{57}
\end{equation*}
$$

s.t.:

$$
\begin{align*}
d W_{t}^{i} & =q W_{t}^{i} d t+\pi_{t}^{i}\left\{\mu d t+\sigma d B_{t}\right\}+\left\{W_{t}^{i}-\pi_{t}^{i}\right\} r d t  \tag{58}\\
& -c_{t}^{i} d t+d G_{t}+d N_{t}^{i}-\tilde{Y}_{t}^{i} d t  \tag{59}\\
W_{0^{+}}^{i} & =W_{0}^{i}-D_{0}  \tag{60}\\
W_{t}^{i} & \geq 0 \text { for all } t>0 \tag{61}
\end{align*}
$$

Problem 5 is almost identical to problem 1, with the main exception that the constraint $c_{t} \geq \xi$ is replaced by the separation requirement (55). Once again, the government introduces

[^19]a fully funded transfer process, but not in an attempt to keep consumption above a minimum standard, but instead with the goal to induce agents who have not experienced idiosyncratic shocks to refrain from using the welfare system. However, the setup does not prohibit agents who have experienced idiosyncratic shocks to use the welfare system, and in general they will.

The link between the two problems is given by the following Lemma.

Lemma $5 d N_{t}^{\mathcal{N I}}=0$ whenever $c_{t}^{\mathcal{N I}} \geq \xi$.
Lemma 5 shows that the constraint (7) is a standard "truth-telling" constraint, which ensures that agents with asset dynamics given by equation (12) (i.e. agents who have not experienced idiosyncratic shocks) do not find it optimal to access the welfare system. Because of this correspondence, problem 1 can be viewed as a limiting case of problem 5 as $\theta$ becomes sufficiently small. The next Proposition formalizes this claim.

Proposition 7 Let $\Omega^{*}\left(W_{0}\right)$ denote the value function of problem 5 and let $\Omega\left(W_{0} ; G_{t}^{*}, D_{0}^{*}\right)$ denote the value of the objective function of problem 5 assuming that the government follows any of the optimal policies $G_{t}^{*}, D_{0}^{*}$ for problem 1. Then $\lim _{\theta \rightarrow 0} \Omega\left(W_{0} ; G_{t}^{*}, D_{0}^{*}\right)=\Omega^{*}\left(W_{0}\right)$.

I conclude this section with a remark on the disutility associated with welfare transfers. With appropriate modifications, the assumption that the agent receives a disutility $\xi^{-\gamma}$ per unit of transfer could be replaced with the assumption that an agent has to incur a fixed disutility $\overline{\bar{\Xi}}$ to enter the welfare system. Subsequent to incurring this disutility the agent can attain a continuation value equal to $\bar{V}$. For instance it could be assumed that after the agent incurs the fixed disutility, the government excludes the agent from further trading, pays costs to observe and pay for her idiosyncratic shock for the remainder of her life, and gives her some consumption path with discounted present value $\bar{V}$ in units of utility. In such an extension, the truth-telling constraint $c_{t} \geq \xi$ would have to be replaced with the constraint $V_{t} \geq \bar{V}-\bar{\Xi}$, and so would equation (24) in problem 2 . Then problem 2 shares several mathematical similarities with problems of one-sided commitment. ${ }^{35}$ The solution of such problems can be expressed in the form (25). Because of this correspondence, it is relatively straightforward to simply repeat all the steps of our analysis for the appropriately redefined optimal process $X_{t}^{*}$ associated with such a modified setup. Having determined the optimal process $X_{t}^{*}$, one can address the core issue of the analysis, namely the "implementation" of

[^20]the resulting optimal consumption process by appropriate income process, along the lines of section 6.

## B Proofs

Proof of proposition 1. Subject to minor modifications, the proof of this proposition is identical to the first theorem of He and Pages (1993) and the reader is referred to that paper for a proof.

Proof of Lemma 1. The proof of this lemma is contained in the proof of proposition 2 (Particularly Lemma 7).

Proof of Proposition 2. The proof of this proposition is established in steps. The following Lemma contains a useful first result.

Lemma 6 Take any $\lambda \in\left(0, \xi^{-\gamma}\right]$ and any process $G_{t}$ and define

$$
\begin{equation*}
\widehat{X}_{t} \equiv \arg \min _{X_{s} \in \mathcal{D}} E_{0}\left(\int_{0}^{\infty} e^{-(\rho+q) s} \max _{c_{s}}\left(\frac{c_{s}^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s} H_{s} X_{s} c_{s}\right) d s+\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(X_{s}-1\right) d G_{s}\right) \tag{62}
\end{equation*}
$$

Then:

$$
\begin{equation*}
\lambda E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\widehat{X}_{s}-1\right) d G_{s}\right)=E_{0} \int_{0}^{\infty} e^{-(\rho+q) s}\left(e^{\rho s} \lambda H_{s} \widehat{X}_{s}\right)^{1-\frac{1}{\gamma}}\left(1-\frac{1}{\widehat{X}_{s}}\right) d s \tag{63}
\end{equation*}
$$

Proof of Lemma 6. Let $\Lambda_{t} \equiv 1-\frac{1}{\widehat{X}_{t}}$. Applying Ito's Lemma to $\Lambda_{t}$, one obtains $d \Lambda_{t} \equiv \frac{d \widehat{X}_{t}}{\left(\widehat{X}_{t}\right)^{2}}$. Hence $\Lambda_{t}$ changes when and only $\widehat{X}_{t}$ changes. By Theorem 1 of He and Pages (1993):

$$
\begin{equation*}
\int_{0}^{\infty}\left[E_{t}\left(\int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right)-E_{t}\left(\int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} c_{s} d s\right)\right] d \widehat{X}_{t}=0 \tag{64}
\end{equation*}
$$

where $c_{s}$ is given explicitly by (25). Plugging (25) into (64), and observing that $\Lambda_{t}$ changes when and only when $\widehat{X}_{t}$ changes implies that

$$
\int_{0}^{\infty}\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}-E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s}\left(e^{\rho s} \lambda H_{s} \widehat{X}_{s}\right)^{-\frac{1}{\gamma}} d s\right) d \Lambda_{t}=0 .
$$

Then, for any admissible $G_{t}$ and $\widehat{X}_{t}$ given by (62)

$$
\lambda E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\widehat{X}_{s}-1\right) d G_{s}\right)=
$$

$$
\begin{align*}
& \lambda E_{0}\left[\int_{0}^{\infty} e^{-q s} H_{s}\left(\widehat{X}_{s}-1\right) d G_{s}-\int_{0}^{\infty}\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right) d \Lambda_{t}\right]  \tag{65}\\
& +\lambda E_{0}\left\{\int_{0}^{\infty} E_{t}\left[\int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s}\left(e^{\rho s} \lambda H_{s} \widehat{X}_{s}\right)^{-\frac{1}{\gamma}} d s\right] d \Lambda_{t}\right\} .
\end{align*}
$$

Next consider the martingale

$$
\begin{equation*}
\mathcal{M}_{t} \equiv E_{t} \int_{0}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}=\int_{0}^{t} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}+E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s} \tag{66}
\end{equation*}
$$

According to the martingale representation theorem, there exists a square integrable $\widetilde{\psi}_{s}$ such that

$$
\begin{equation*}
\mathcal{M}_{t}=\mathcal{M}_{0}+\int_{0}^{t} \widetilde{\psi}_{s} d B_{s} \tag{67}
\end{equation*}
$$

Combining (66) and (67) gives

$$
\begin{aligned}
d\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right) & =d \mathcal{M}_{t}-\widehat{X}_{t} e^{-q t} H_{t} d G_{t} \\
& =\widetilde{\psi}_{t} d B_{t}-\widehat{X}_{t} e^{-q t} H_{t} d G_{t}
\end{aligned}
$$

Now, fixing an arbitrary $\varepsilon>0$, letting $\tau^{\varepsilon}$ be the first time $t$ such that $\left|\Lambda_{t}\right| \geq \frac{1}{\varepsilon}$, applying integration by parts and using the fact that $\Lambda_{0}=0$, gives

$$
\begin{aligned}
-E_{0} \int_{0}^{T \wedge \tau^{\varepsilon}}\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right) d \Lambda_{t} & =-E_{0} \int_{0}^{T \wedge \tau^{\varepsilon}} \Lambda_{s} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}+E_{0} \int_{0}^{T \wedge \tau^{\varepsilon}} \Lambda_{s} \widetilde{\psi}_{s} d B_{s} \\
& -E_{0}\left[\Lambda_{T \wedge \tau^{\varepsilon}}\left(E_{T \wedge \tau^{\varepsilon}} \int_{T \wedge \tau^{\varepsilon}}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right)\right]
\end{aligned}
$$

Since $\psi_{s}$ is square integrable and $\left|\Lambda_{s}\right|$ is bounded in $\left[0, \frac{1}{\varepsilon}\right]$ the second term on the right hand side of the above expression is 0 . Also note that

$$
\begin{equation*}
-E_{0}\left[\Lambda_{T \wedge \tau^{\varepsilon}}\left(E_{T \wedge \tau^{\varepsilon}} \int_{T \wedge \tau^{\varepsilon}}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right)\right]=-E_{0}\left[\widehat{X}_{T \wedge \tau^{\varepsilon}} \Lambda_{T \wedge \tau^{\varepsilon}} J\right], \tag{68}
\end{equation*}
$$

where

$$
\begin{equation*}
J \equiv\left(E_{T \wedge \tau^{\varepsilon}} \int_{T \wedge \tau^{\varepsilon}}^{\infty} \frac{\widehat{X}_{s}}{\widehat{X}_{T \wedge \tau^{\varepsilon}}} e^{-q s} H_{s} d G_{s}\right) \leq E_{T \wedge \tau^{\varepsilon}} \int_{T \wedge \tau^{\varepsilon}}^{\infty} e^{-q s} H_{s} d G_{s}, \tag{69}
\end{equation*}
$$

since $\widehat{X}_{t}$ is non-increasing. Combining (69) with (68) and noting that $0<\widehat{X}_{t} \leq 1$,

$$
\begin{equation*}
-E_{0}\left[\widehat{X}_{T \wedge \tau^{\varepsilon}} \Lambda_{T \wedge \tau^{\varepsilon}} J\right]=E_{0}\left[\left(1-\widehat{X}_{T \wedge \tau^{\varepsilon}}\right) J\right] \leq E_{T \wedge \tau^{\varepsilon}} \int_{T \wedge \tau^{\varepsilon}}^{\infty} e^{-q s} H_{s} d G_{s} \tag{70}
\end{equation*}
$$

Given that $E \int_{0}^{\infty} e^{-q s} H_{s} d G_{s}<\infty$ it follows that

$$
\begin{equation*}
E_{T \wedge \tau^{\varepsilon}} \int_{T \wedge \tau^{\varepsilon}}^{\infty} e^{-q s} H_{s} d G_{s} \rightarrow 0 \tag{71}
\end{equation*}
$$

as $\varepsilon \rightarrow 0, T \rightarrow \infty$. This leads to the inequalities:

$$
\begin{aligned}
-E_{0} \int_{0}^{\infty}\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right) d \Lambda_{t} & \geq-E_{0} \int_{0}^{T \wedge \tau^{\varepsilon}}\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right) d \Lambda_{t} \\
& \geq-E_{0} \int_{0}^{T \wedge \tau^{\varepsilon}} \Lambda_{s} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}
\end{aligned}
$$

Letting $\varepsilon \rightarrow 0, T \rightarrow \infty$, using the monotone convergence theorem, and using (70) and (71), gives

$$
\begin{equation*}
-\int_{0}^{\infty}\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right) d \Lambda_{t}=-E_{0} \int_{0}^{\infty} \Lambda_{s} \widehat{X}_{s} e^{-q s} H_{s} d G_{s} \tag{72}
\end{equation*}
$$

Using (72) and the definition of $\Lambda_{t}$ gives

$$
\begin{aligned}
& \lambda E_{0}\left[\int_{0}^{\infty} e^{-q s} H_{s}\left(\widehat{X}_{s}-1\right) d G_{s}-\int_{0}^{\infty}\left(E_{t} \int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s} d G_{s}\right) d \Lambda_{t}\right]= \\
& =E_{0}\left[\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(\widehat{X}_{s}-1\right) d G_{s}-\lambda \int_{0}^{\infty} e^{-q s} H_{s} \widehat{X}_{s} \Lambda_{s} d G_{s}\right]=0 .
\end{aligned}
$$

Returning now to (65) and using the above equation yields

$$
\begin{align*}
\lambda E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\widehat{X}_{s}-1\right) d G_{s}\right) & =\lambda E_{0}\left\{\int_{0}^{\infty} E_{t}\left[\int_{t}^{\infty} \widehat{X}_{s} e^{-q s} H_{s}\left(e^{\rho s} \lambda H_{s} \widehat{X}_{s}\right)^{-\frac{1}{\gamma}} d s\right] d \Lambda_{t}\right\}  \tag{73}\\
& =E_{0}\left[\int_{0}^{\infty} e^{-(\rho+q) t}\left(e^{\rho t} \lambda H_{t} \widehat{X}_{s}\right)^{1-\frac{1}{\gamma}} \Lambda_{t} d t\right] \tag{74}
\end{align*}
$$

where (74) follows from a similar integration by parts argument as the one in equations (66)-(72).

The next Lemma uses Lemma 6 to prove (26).
Lemma 7 For all admissible processes $G_{t} \in \mathcal{G}$ :
$\max _{G_{t} \in \mathcal{G}} V\left(W_{0}\right) \leq \min _{\lambda \in\left(0, \xi^{-\gamma]}\right.}\left[E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{1-\frac{1}{\gamma}}}{1-\gamma} d s-\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s+\lambda W_{0}\right)\right]$

Proof of Lemma 7. Proposition 1 along with Lemma 6 implies that for any admissible process
$G_{t}$ there exists a $\lambda^{G}>0$ and a decreasing process $X_{t}^{G} \in \mathcal{D}$ that minimizes (19) such that

$$
\begin{align*}
V\left(W_{0}\right) & =E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \max _{c_{s}}\left(\frac{c_{s}^{1-\gamma}}{1-\gamma}-\lambda^{G} e^{\rho s} H_{s} X_{s}^{G} c_{s}\right) d s+\lambda^{G} \int_{0}^{\infty} e^{-q s} H_{s}\left(X_{s}^{G}-1\right) d G_{s}\right)+\lambda^{G} W_{0} \\
& =E \int_{0}^{\infty} e^{-(\rho+q) s}\left(\frac{\left(e^{\rho s} \lambda^{G} H_{s} X_{s}^{G}\right)^{1-\frac{1}{\gamma}}}{1-\gamma}-\lambda^{G} e^{\rho s} H_{s}\left(e^{\rho s} \lambda^{G} H_{s} X_{s}^{G}\right)^{-\frac{1}{\gamma}}\right) d s+\lambda^{G} W_{0} . \tag{76}
\end{align*}
$$

Moreover, since the process $G_{t}$ enforces $c_{t} \geq \xi$, equation (18) implies that $\lambda^{G} \leq \xi^{-\gamma}$. Next take an arbitrary $\lambda>0$. Since $c_{t}=\left(e^{\rho t} \lambda^{G} H_{t} X_{t}^{G}\right)^{-\frac{1}{\gamma}}$ is an optimal consumption process, it exhausts the "budget constraint" of the consumer so that

$$
E \int_{0}^{\infty} e^{-(\rho+q) s} e^{\rho s} H_{s}\left(e^{\rho s} \lambda^{G} H_{s} X_{s}^{G}\right)^{-\frac{1}{\gamma}} d s=W_{0}-D_{0}+E \int_{0}^{\infty} e^{-q s} H_{s} d G_{s} .
$$

Using (10), this implies that $E \int_{0}^{\infty} e^{-(\rho+q) s} e^{\rho s} H_{s}\left(e^{\rho s} \lambda^{G} H_{s} X_{s}^{G}\right)^{-\frac{1}{\gamma}}=W_{0}$. This furthermore implies that (76) can be rewritten as

$$
\begin{equation*}
V\left(W_{0}\right)=E \int_{0}^{\infty} e^{-(\rho+q) s}\left(\frac{\left(e^{\rho s} \lambda^{G} H_{s} X_{s}^{G}\right)^{1-\frac{1}{\gamma}}}{1-\gamma}-\lambda e^{\rho s} H_{s}\left(e^{\rho s} \lambda^{G} H_{s} X_{s}^{G}\right)^{-\frac{1}{\gamma}}\right) d s+\lambda W_{0} . \tag{77}
\end{equation*}
$$

Next define $X_{t}^{*}$ as in equation (30), and let the process $N_{t}$ be given as $N_{t} \equiv \frac{\lambda^{G}}{\lambda} \frac{X_{t}^{G}}{X_{t}^{*}}$. Using $N_{t}$, one can rewrite equation (77) as

$$
\begin{equation*}
V\left(W_{0}\right)=E \int_{0}^{\infty} e^{-(\rho+q) s}\left(\frac{\left(e^{\rho s} \lambda H_{s} X_{s}^{*} N_{s}\right)^{1-\frac{1}{\gamma}}}{1-\gamma}-\lambda e^{\rho s} H_{s}\left(e^{\rho s} \lambda H_{s} X_{s}^{*} N_{s}\right)^{-\frac{1}{\gamma}}\right) d s+\lambda W_{0} . \tag{78}
\end{equation*}
$$

Since $\lambda^{G} X_{t}^{G}$ is a decreasing process that starts at $\lambda^{G}$ and always stays below $\xi^{-\gamma}$, the Skorohod equation ${ }^{36}$ implies that there exists another decreasing process $\lambda^{G} X_{t}^{* G}$ that also starts at $\lambda^{G}$ and stays below $\xi^{-\gamma}$, with the property

$$
\begin{equation*}
\lambda^{G} X_{t}^{G} \leq \lambda^{G} X_{t}^{* G} \tag{79}
\end{equation*}
$$

This process is given by $X_{t}^{* G}=\min \left[1, \frac{\xi^{-\gamma} / \lambda^{G}}{\max _{0 \leq s \leq t}\left(e^{\rho s} H_{s}\right)}\right]$. Note that $X_{t}^{* G}$ is identical to $X_{t}^{*}$ with the exception that $\lambda$ replaces $\lambda^{G}$. Using (79) and the definition of $N_{t}$ yields

$$
\begin{equation*}
N_{t}=\frac{\lambda^{G}}{\lambda} \frac{X_{t}^{G}}{X_{t}^{*}} \leq \frac{\lambda^{G}}{\lambda} \frac{X_{t}^{* G}}{X_{t}^{*}} \tag{80}
\end{equation*}
$$

[^21]Using (80) and (78) leads to

$$
\begin{equation*}
V\left(W_{0}\right) \leq E \int_{0}^{\infty} e^{-(\rho+q) s} A(s) d s+\lambda W_{0} \tag{81}
\end{equation*}
$$

where

$$
\begin{equation*}
A(s) \equiv \max _{N_{s} \leq Q_{s}}(\widetilde{A}(s)) \tag{82}
\end{equation*}
$$

and $\widetilde{A}(s)$ is defined as $\widetilde{A}(s) \equiv \frac{\left(e^{\rho s} \lambda H_{s} X_{s}^{*} N_{s}\right)^{1-\frac{1}{\gamma}}}{1-\gamma}-\lambda e^{\rho s} H_{s}\left(e^{\rho s} \lambda H_{s} X_{s}^{*} N_{s}\right)^{-\frac{1}{\gamma}}$, while $Q_{s} \equiv \max \left[1, \frac{\lambda^{G}}{\lambda} \frac{X_{X_{s}^{*}}}{X_{s}^{*}}\right]$. To study the maximization problem of equation (82) it is useful to compute the derivative of $\widetilde{A}_{s}$ with respect to $N_{s}$. Performing this computation and combining terms gives

$$
\begin{equation*}
\frac{\partial \widetilde{A}_{s}}{\partial N_{s}}=-\frac{1}{\gamma}\left(e^{\rho s} \lambda H_{s} X_{s}^{*} N_{s}\right)^{1-\frac{1}{\gamma}} N_{s}^{-1}\left(1-\frac{1}{N_{s} X_{s}^{*}}\right) \tag{83}
\end{equation*}
$$

At this stage it is useful to consider two cases separately. The first case is $\lambda>\lambda^{G}$. In this case, it is straightforward to show that $Q_{s}=1$. Hence in maximizing $\widetilde{A}(s)$, one can constrain attention to values of $N_{s} \leq 1$. An examination of (83) reveals that $\frac{\partial \widetilde{A}(s)}{\partial N_{s}} \geq 0$ for all $N_{s} \leq 1$ and all $X_{s}^{*}$, since $X_{s}^{*} \leq 1$. Hence the solution to (82) is $N_{s}=1$ when $\lambda>\lambda^{G}$.

In the case where $\lambda<\lambda^{G}$ it is also true that the optimal $N_{s}$ in (82) is equal to one. To see this, observe that

$$
Q_{s}= \begin{cases}\frac{\lambda^{G}}{\lambda} \frac{X_{s}^{* G}}{X_{s}^{*}} & \text { when } X_{s}^{*}=1 \\ 1 & \text { when } X_{s}^{*}<1\end{cases}
$$

Using this observation in (83) reveals that the optimal choice for $N_{s}$ is always equal to 1. ${ }^{37}$
The above reasoning shows that the optimal solution of (82) is given by $N_{s}=1$. Returning to (81), this implies that

$$
V\left(W_{0}\right) \leq E \int_{0}^{\infty} e^{-(\rho+q) s}\left(\frac{\left(e^{\rho s} \lambda H_{s} X_{s}^{*}\right)^{1-\frac{1}{\gamma}}}{1-\gamma} d s-\lambda e^{\rho s} H_{s}\left(e^{\rho s} \lambda H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right)+\lambda W_{0} .
$$

Since this bound holds for arbitrary $\lambda \in\left(0, \xi^{-\gamma}\right]$ and arbitrary $G_{t} \in \mathcal{G}$, it also holds for the $\lambda \in\left(0, \xi^{-\gamma}\right]$ that minimizes the right hand side of the above equation and the $G_{t} \in \mathcal{G}$ that maximizes the right hand side. Hence (75) follows.

The next part of the proof of Proposition 2 is to show that equation (31) holds. A first step is to show that (31) provides an upper bound to $J\left(W_{0}\right)$.

[^22]Lemma 8 The value function of problem 2 is bounded above by
$J\left(W_{0}\right) \leq \min _{\lambda \in\left(0, \xi^{-\gamma}\right]}\left[E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{1-\frac{1}{\gamma}}}{1-\gamma} d s-\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s+\lambda W_{0}\right)\right]$

Proof of Lemma 8. The proof of this Lemma follows identical steps to the proof of the previous Lemma. To see this, take an arbitrary triplet $<\widehat{\lambda}, X_{t}, c_{t}>$ that satisfies equations (23)(25) of Problem 2. Then for any $\lambda>0$, one obtains

$$
J\left(W_{0}\right) \leq E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\left(\widehat{\lambda} e^{\rho s} H_{s} X_{s}\right)^{1-\frac{1}{\gamma}}}{1-\gamma}-\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(\widehat{\lambda} e^{\rho s} H_{s} X_{s}\right)^{-\frac{1}{\gamma}}+\lambda W_{0}\right)
$$

Notice that this equation is identical to equation (77), with the exception that $\lambda^{G}$ is replaced by $\widehat{\lambda}$ and $X_{t}^{G}$ is replaced by $X_{t}$. Since the equations following (77) hold for any $\lambda^{G}, X_{t}^{G}$ they also hold for $\widehat{\lambda}, X_{t}$. Accordingly, by repeating the same steps, one can arrive at (84).

The next step in the proof of the proposition is to show that the inequality in (84) holds with equality for the optimal policy. The following Lemma presents a step in this direction.

Lemma 9 Let $F(\lambda)$ be given by

$$
\begin{equation*}
F(\lambda)=E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{1-\frac{1}{\gamma}}}{1-\gamma} d s-\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right) \tag{85}
\end{equation*}
$$

Then

$$
\begin{equation*}
F(\lambda)=-\frac{K \xi^{1-\gamma}}{\gamma \phi(\phi-1)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi}+K \frac{\gamma}{1-\gamma} \lambda^{1-\frac{1}{\gamma}} \tag{86}
\end{equation*}
$$

Assume moreover that (29) is met. Then

$$
\begin{equation*}
\min _{\lambda \in\left(0, \xi^{-\gamma}\right]}\left[F(\lambda)+\lambda W_{0}\right]=\min _{\lambda>0}\left[F(\lambda)+\lambda W_{0}\right] \tag{87}
\end{equation*}
$$

and (84) can be rewritten as $J\left(W_{0}\right) \leq \min _{\lambda>0}\left[F(\lambda)+\lambda W_{0}\right]$. Moreover, letting $\lambda^{*}$ be given as $\lambda^{*} \equiv \arg \min _{\lambda>0}\left[F(\lambda)+\lambda W_{0}\right]$ implies that $E_{0}\left[\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda^{*} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}}\right]=W_{0}$, and accordingly $c_{s}^{*}=\left(\lambda^{*} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}}$ is a feasible consumption process for problem 2.

Proof of Lemma 9. To save notation, let

$$
\begin{equation*}
Z_{t} \equiv \lambda e^{\rho t} H_{t} X_{t}^{*} \tag{88}
\end{equation*}
$$

and note that $Z_{0}=\lambda$, and that $Z_{t} \in\left(0, \xi^{-\gamma}\right]$ by the definition of $X_{t}^{*}$ in equation (30). Equation
(85) can now be rewritten as

$$
\begin{equation*}
F(\lambda)=E\left[\int_{0}^{\infty} e^{-(\rho+q) s} \frac{1}{1-\gamma}\left(Z_{s}\right)^{1-\frac{1}{\gamma}} d s-\int_{0}^{\infty} e^{-(\rho+q) s} \frac{Z_{s}^{1-\frac{1}{\gamma}}}{X_{s}^{*}} d s\right] . \tag{89}
\end{equation*}
$$

It will be convenient to compute the two terms inside equation (89) separately. Define first

$$
\begin{equation*}
G\left(Z_{t}\right) \equiv E\left[\left.\int_{t}^{\infty} e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma}\left(Z_{s}\right)^{1-\frac{1}{\gamma}} d s \right\rvert\, Z_{t}\right] . \tag{90}
\end{equation*}
$$

To compute $G\left(Z_{t}\right)$, it is easiest to let $\tau^{\varepsilon}$ be the first hitting time of $Z_{t}$ to the level $\varepsilon>0$, namely $\tau^{\varepsilon} \equiv \inf _{s \geq t}\left\{Z_{s}=\varepsilon\right\}$, and then compute the expression:

$$
\begin{equation*}
G^{\varepsilon}\left(Z_{t}\right)=E\left[\left.\int_{t}^{\tau^{\varepsilon}} e^{-(\rho+q) s} \frac{1}{1-\gamma}\left(Z_{s}\right)^{1-\frac{1}{\gamma}} d s \right\rvert\, Z_{t}\right] . \tag{91}
\end{equation*}
$$

To compute (91), apply first Ito's Lemma to (88) to obtain $\frac{d Z_{t}}{Z_{t}}=(\rho-r) d t-\kappa d B_{t}+\frac{d X_{t}^{*}}{X_{t}^{*}}$. Next, construct a function $G^{\varepsilon}(Z)$ that satisfies the ODE

$$
\begin{equation*}
\frac{\kappa^{2}}{2} G_{Z Z}^{\varepsilon} Z^{2}+G_{Z}^{\varepsilon} Z(\rho-r)-(\rho+q) G^{\varepsilon}+\frac{1}{1-\gamma}(Z)^{1-\frac{1}{\gamma}}=0 \tag{92}
\end{equation*}
$$

subject to the boundary conditions $G_{Z}^{\varepsilon}\left(\xi^{-\gamma}\right)=0, G^{\varepsilon}(\varepsilon)=0$.
Equation (92) is a linear ordinary differential equation with general solution

$$
G^{\varepsilon}(Z)=C_{1} Z^{\phi^{-}}+C_{2} Z^{\phi}+K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}},
$$

where $C_{1}, C_{2}$ are arbitrary constants, $K$ is given in equation (28), $\phi>0$ in (27), and $\phi^{-}$is given by

$$
\begin{equation*}
\phi^{-} \equiv \frac{-\left(\rho-r-\frac{\kappa^{2}}{2}\right)-\sqrt{\left(\rho-r-\frac{\kappa^{2}}{2}\right)^{2}+2(\rho+q) \kappa^{2}}}{\kappa^{2}}<0 \tag{93}
\end{equation*}
$$

To satisfy the two boundary conditions $G_{Z}^{\varepsilon}\left(\xi^{-\gamma}\right)=0, G^{\varepsilon}(\varepsilon)=0$, the constants $C_{1}$ and $C_{2}$ must be chosen so that

$$
\phi^{-} C_{1}\left(\xi^{-\gamma}\right)^{\phi^{-}}+\phi C_{2}\left(\xi^{-\gamma}\right)^{\phi}-\frac{1}{\gamma} K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}}=0, C_{1} \varepsilon^{\phi^{-}}+C_{2} \varepsilon^{\phi}+K \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}}=0 .
$$

Solving this system yields:

$$
C_{2}=\frac{K\left[\frac{1}{\gamma \phi^{-}}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\phi^{-}} \varepsilon^{\phi^{-}}+\frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}}\right]}{\frac{\phi}{\phi^{-}}\left(\xi^{-\gamma}\right)^{\phi-\phi^{-}} \varepsilon^{\phi^{-}}-\varepsilon^{\phi}}, C_{1}=-C_{2} \varepsilon^{\phi-\phi^{-}}-K \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}-\phi^{-}} .
$$

It remains now to verify that $G^{\varepsilon}\left(Z_{t}\right)$ satisfies (91). To this end, apply Ito's Lemma to $e^{-(\rho+q) t} G^{\varepsilon}\left(Z_{t}\right)$ to obtain for any time $T \wedge \tau^{\varepsilon}$

$$
\begin{aligned}
e^{-(\rho+q) T} G^{\varepsilon}\left(Z_{T \wedge \tau}\right)-e^{-(\rho+q) t} G^{\varepsilon}\left(Z_{t}\right) & =\int_{t}^{T \wedge \tau^{\varepsilon}}\left(\frac{\kappa^{2}}{2} G_{Z Z}^{\varepsilon} Z_{s}^{2}+G_{Z}^{\varepsilon} Z_{s}(\rho-r)-(\rho+q) G^{\varepsilon}\right) e^{-(\rho+q) s} d s \\
& -\int_{t}^{T \wedge \tau^{\varepsilon}} e^{-(\rho+q) s} \kappa G_{Z}^{\varepsilon} Z_{s} d B_{s}+\int_{t}^{T \wedge \tau^{\varepsilon}} e^{-(\rho+q) s} G_{Z}^{\varepsilon}\left(\xi^{-\gamma}\right) \xi^{-\gamma} \frac{d X_{s}^{*}}{X_{s}^{*}} .
\end{aligned}
$$

Using (92) inside the first term on the right hand side of the above equation along with $G_{Z}^{\varepsilon}\left(\xi^{-\gamma}\right)=0$ inside the third term, letting $T \rightarrow \infty$ along with $G^{\varepsilon}(\varepsilon)=0$, and using the monotone convergence theorem gives

$$
\begin{equation*}
G^{\varepsilon}\left(Z_{t}\right)=E_{t}\left[\int_{t}^{\tau^{\varepsilon}} e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma}\left(Z_{s}\right)^{1-\frac{1}{\gamma}} d s+\int_{t}^{\tau^{\varepsilon}} e^{-(\rho+q)(s-t)} \kappa G_{Z}^{\varepsilon} Z_{s} d B_{s}\right] . \tag{94}
\end{equation*}
$$

Since $G_{Z}^{\varepsilon} Z$ is bounded between $t$ and $\tau^{\varepsilon}$, the second term in the above expression is a martingale and hence (113) follows. Next, letting $\varepsilon \rightarrow 0$, it is straightforward to show that

$$
C_{2}=\frac{K\left[\frac{1}{\gamma \phi^{-}}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\phi^{-}}-\frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}-\phi^{-}}\right]}{\frac{\phi}{\phi^{-}}\left(\xi^{-\gamma}\right)^{\phi-\phi^{-}}-\varepsilon^{\phi-\phi^{-}}} \rightarrow K \frac{1}{\gamma \phi}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\phi},
$$

since $\varepsilon^{\phi-\phi^{-}} \rightarrow 0$ and $\varepsilon^{1-\frac{1}{\gamma}-\phi^{-}} \rightarrow 0$. By a similar argument it is easy to show that $C_{1} \rightarrow 0$ and hence:

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} G^{\varepsilon}(Z)=G(Z)=\frac{1}{\phi} \frac{1}{\gamma} K \xi^{1-\gamma}\left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi}+K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}} \tag{95}
\end{equation*}
$$

Equation (90) follows as a consequence of the monotone convergence theorem.
It remains to compute the expression

$$
\begin{equation*}
N\left(Z_{t}, X_{t}^{*}\right)=E_{t}\left(\int_{t}^{\infty} e^{-(\rho+q)(s-t)} \frac{Z_{s}^{1-\frac{1}{\gamma}}}{X_{s}^{*}} d s\right) \tag{96}
\end{equation*}
$$

Following similar steps as for $G\left(Z_{t}\right), N\left(Z, X^{*}\right)$ is given by

$$
\begin{equation*}
N\left(Z, X^{*}\right)=\frac{1}{(\phi-1)} \frac{1}{\gamma} \frac{K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}}}{X^{*}}\left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi}+K \frac{Z^{1-\frac{1}{\gamma}}}{X^{*}} . \tag{97}
\end{equation*}
$$

It is now possible to compute $F(\lambda)$ which is given by

$$
\begin{equation*}
F(\lambda)=G(\lambda)-N(\lambda, 1)==-\frac{K \xi^{1-\gamma}}{\gamma \phi(\phi-1)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi}+K \frac{\gamma}{1-\gamma} \lambda^{1-\frac{1}{\gamma}} . \tag{98}
\end{equation*}
$$

To show the second part of the proposition, observe that $(96),(88)$ and (97) imply that

$$
\begin{align*}
\frac{N(\lambda, 1)}{\lambda} & =\frac{1}{\lambda} E_{0}\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{Z_{s}^{1-\frac{1}{\gamma}}}{X_{s}^{*}} d s\right)=E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right)= \\
& =\frac{K \xi^{1-\gamma}}{(\phi-1)} \frac{1}{\gamma}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} \frac{1}{\lambda}+K \lambda^{-\frac{1}{\gamma}} \tag{99}
\end{align*}
$$

Moreover, computing $F^{\prime}(\lambda)$ in (98) yields

$$
\begin{equation*}
F^{\prime}(\lambda)=-\frac{K \xi^{1-\gamma}}{(\phi-1)} \frac{1}{\gamma}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} \frac{1}{\lambda}-K \lambda^{-\frac{1}{\gamma}} \tag{100}
\end{equation*}
$$

Combining (99) and (100) leads to

$$
\begin{equation*}
F^{\prime}(\lambda)=-\frac{N(\lambda, 1)}{\lambda}=-E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right) \tag{101}
\end{equation*}
$$

Using the formula for $F(\lambda)$, equation (84) can be expressed as $\min _{\lambda \in\left(0, \xi^{-\gamma}\right]}\left\{F(\lambda)+\lambda W_{0}\right\}$, which leads to the first order condition $F^{\prime}\left(\lambda^{*}\right)=-W_{0}$. Using (101) leads to

$$
W_{0}=E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda^{*} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right)=E_{0}\left(\int_{0}^{\infty} e^{-q s} H_{s} c_{s}^{*} d s\right)
$$

This last equation implies that $\lambda^{*}, X_{t}^{*}$ and the associated consumption process $c_{t}^{*}=\left(\lambda^{*} e^{\rho t} H_{t} X_{t}^{*}\right)^{-\frac{1}{\gamma}}$ satisfy (23) and (25). To show that the choice $\left\langle\lambda^{*}, X_{t}^{*}, c_{t}^{*}\right\rangle$ constitutes a feasible triplet, it remains to show that it also satisfies (24). By construction of $X_{t}^{*}$ this will be the case as long as $\lambda^{*}<\xi^{-\gamma}$. This will indeed be the case as long as $W_{0}$ satisfies (29). To see this, note that $\xi^{-\gamma}$ is the unique solution of $F^{\prime}\left(\lambda^{*}\right)=-W_{0}$, when $W_{0}$ is given by $W_{0}=\frac{\frac{1}{\gamma}+\phi-1}{\phi-1} K \xi$. Moreover, equation (100) implies that:

$$
\begin{align*}
F^{\prime \prime}(\lambda) & =-K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}} \frac{1}{\gamma}\left(\frac{1}{\xi^{-\gamma}}\right)^{\phi} \lambda^{\phi-2}+\frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma}-1} \\
& =\frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma}-1}\left[1-\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi+\frac{1}{\gamma}-1}\right]>0 \tag{102}
\end{align*}
$$

The above equation shows that $F^{\prime}(\lambda)$ is an increasing function of $\lambda$ for $0<\lambda<\xi^{-\gamma}$ and hence the solution $\lambda^{*}$ of equation $F^{\prime}\left(\lambda^{*}\right)=-W_{0}$ is a decreasing function of $W_{0}$. Hence, as long as $W_{0}$ satisfies (29), then $\lambda^{*}<\xi^{-\gamma}$. Since the interior solution $\lambda^{*}$ is smaller than $\xi^{-\gamma}$, equation (87) follows.

Combining the above Lemma with (84) implies that

$$
\begin{aligned}
J\left(W_{0}\right) & \leq \min _{\lambda>0}\left[F(\lambda)+\lambda W_{0}\right]=F\left(\lambda^{*}\right)+\lambda^{*} W_{0}= \\
& =E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\left(\left(\lambda^{*} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}}\right)^{1-\gamma}}{1-\gamma} d s\right) \\
& =E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\left(c_{s}^{*}\right)^{1-\gamma}}{1-\gamma} d s\right) \leq J\left(W_{0}\right) .
\end{aligned}
$$

The last inequality follows because $c_{s}^{*}=\left(\lambda^{*} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}}$ is a feasible consumption process for problem for problem 2 and $J\left(W_{0}\right)$ is the value function of the problem. The above three lines imply that equation (84) holds with equality as long as one chooses the optimal solution in the statement of the proposition. This concludes the proof of Proposition 2.

Proof of Proposition 3. The proof of this Proposition is just a special case of Section 6 in He and Pages (1993) and hence I give only a sketch and refer the reader to He and Pages (1993) for details. To start, define

$$
\begin{equation*}
\widetilde{V}(\lambda)=\min _{X_{s} \in \mathcal{D}} E\left[\int_{0}^{\infty} e^{-(\rho+q) s} \max _{c_{s}}\left(\frac{c_{s}^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s} H_{s} X_{s} c_{s}\right) d s+\lambda \int_{0}^{\infty} e^{-q s} H_{s} X_{s} y_{0} d s\right] . \tag{103}
\end{equation*}
$$

By equation (10) and equation (19) of Proposition 1

$$
\begin{equation*}
V\left(W_{0}\right)=\min _{\lambda>0}\left[\widetilde{V}(\lambda)+\lambda\left(W_{0}-\frac{y_{0}}{r+q}\right)\right], \tag{104}
\end{equation*}
$$

since $y_{0} E \int_{0}^{\infty} H_{s} d s=\frac{y_{0}}{r}$. Next, for an arbitrary decreasing process $X_{t}$ let $Z_{t}$ be defined as $Z_{t} \equiv$ $\lambda e^{\rho s} H_{s} X_{s}$, and note that $Z_{0}=\lambda$. Applying Ito's Lemma to $Z_{t}$ gives:

$$
\begin{equation*}
\frac{d Z_{t}}{Z_{t}}=(\rho-r) d t-\kappa d B_{t}+\frac{d X_{t}}{X_{t}} \tag{105}
\end{equation*}
$$

With this definition of $Z_{t}$ one can solve the maximization problem inside (103) and rewrite $\widetilde{V}(\lambda)$ as

$$
\begin{equation*}
\widetilde{V}\left(Z_{0}\right)=\min _{X_{s} \in \mathcal{D}} E\left[\int_{0}^{\infty} e^{-(\rho+q) s}\left(\frac{\gamma}{1-\gamma} Z_{s}^{1-\frac{1}{\gamma}}+y_{0} Z_{s}\right) d s\right] \tag{106}
\end{equation*}
$$

From this point on, one can use similar arguments to He and Pages (1993), and treat (106) as a singular stochastic control problem over the set of decreasing processes $X_{t}$. As He and Pages (1993) show, the optimal solution is to always decrease $X_{t}$ appropriately, so as to keep $Z_{t}$ in the interval $(0, \bar{Z}] . \bar{Z}$ is a free boundary that is determined next.

Using this conjecture for the optimal policy one can now proceed as He and Pages (1993) to
establish that $\widetilde{V}(Z)$ satisfies the ordinary differential equation:

$$
\frac{\kappa^{2}}{2} \widetilde{V}_{Z Z} Z^{2}+(\rho-r) \widetilde{V}_{Z} Z-(\rho+q) \widetilde{V}+\frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}}+y_{0} Z=0 \text { for all } Z \in(0, \bar{Z}] .
$$

The general solution to this equation is

$$
\begin{equation*}
\tilde{V}(Z)=C_{1} Z^{\phi}+C_{2} Z^{\phi^{-}}+K \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}}+\frac{y_{0}}{r+q} Z, \tag{107}
\end{equation*}
$$

where $K$ is given in (28), $\phi$ in (27) and $\phi^{-}$in (93) and $C_{1}, C_{2}$ are arbitrary constants. By arguments similar to He and Pages (1993), one can set $C_{2}=0$ (since $\phi^{-}<0$ ). Hence it remains to determine $C_{1}$ and the free boundary $\bar{Z}$. As most singular stochastic control problems, one can employ a "smooth pasting" and "high contact" principle, namely by determining $C_{1}$ and $\bar{Z}$ so that $\widetilde{V}_{Z}(\bar{Z})=0$, $\widetilde{V}_{Z Z}(\bar{Z})=0$. Using the "smooth pasting" and "high contact" conditions, along with the general solution in (107) and $C_{2}=0$, one can solve for $C_{1}$ and $\bar{Z}$ to obtain

$$
\begin{align*}
\bar{Z}^{-\frac{1}{\gamma}} & =\frac{1}{K} \frac{y_{0}}{r+q}\left(\frac{\phi-1}{\frac{1}{\gamma}+\phi-1}\right)  \tag{108}\\
C_{1} & =-\frac{\frac{1}{\gamma} \frac{y_{0}}{r+q}}{\phi \bar{Z}^{\phi-1}\left[\frac{1}{\gamma}+\phi-1\right]} \tag{109}
\end{align*}
$$

The next steps to verify that the conjectured policy is indeed optimal are identical to He and Pages (1993) and are left out.

To conclude the proof, note that sofar the calculations were true for an arbitrary $y_{0}$. To determine the $y_{0}$ that will safeguard that $c_{t} \geq \xi$ observe that $c_{t}=Z^{-\frac{1}{\gamma}}$ by equation (18). Since the optimal policy is to control $X_{t}$ so as to "keep" $Z_{t}$ in the interval $(0, \bar{Z}]$ it follows that the minimum level of consumption is given by $\bar{Z}^{-\frac{1}{\gamma}}$. Hence, in order to guarantee condition $c_{t} \geq \xi$ it suffices to determine $y_{0}$ so that

$$
\xi=\bar{Z}^{-\frac{1}{\gamma}}=\frac{1}{K} \frac{y_{0}}{r+q}\left(\frac{\phi-1}{\frac{1}{\gamma}+\phi-1}\right) .
$$

Solving for $y_{0}$ gives

$$
y_{0}=\xi(r+q) K \frac{\frac{1}{\gamma}+\phi-1}{\phi-1} .
$$

One can now substitute that level of $y_{0}$ into (109), (108) and use the resulting expressions to obtain from (107) the following expression for $\widetilde{V}(Z)$ :

$$
\widetilde{V}(Z)=-\frac{K \xi^{1-\gamma}}{\gamma \phi(\phi-1)}\left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi}+K \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}}+\frac{y_{0}}{r+q} Z .
$$

Evaluating this expression at $Z_{0}=\lambda$ and using equation (104) gives equation (32), which shows
that the "constant income" policy of the current proposition attains the upper bound of Proposition 2.

Proof of Lemma 2. First note that $\lim _{\gamma \rightarrow \infty}\left(\frac{y_{0}}{\xi}\right)=1$. To show the result, it suffices to show that $\frac{d\left(\frac{y_{0}}{\xi}\right)}{d \gamma}<0$. Differentiating $\frac{y_{0}}{\xi}$ with respect to $\gamma$ gives

$$
\frac{d\left(\frac{y_{0}}{\xi}\right)}{d \gamma}=\frac{(r+q)}{\phi-1} \frac{B}{\left(\frac{\gamma-1}{\gamma} \frac{\kappa^{2}}{2}+\gamma(r+q)+\rho-r\right)^{2}}
$$

where

$$
\begin{aligned}
& B \equiv(\phi-1)(\rho-r)-(r+q)+(\phi-1) \frac{\kappa^{2}}{2} \\
& -(\phi-1) \frac{1}{\gamma} \frac{\kappa^{2}}{2}-[\gamma(\phi-1)+1] \frac{1}{\gamma^{2}} \frac{\kappa^{2}}{2} .
\end{aligned}
$$

Since $\phi>1$ and $r+q>0$, it follows that $\frac{d\left(\frac{y_{0}}{\xi}\right)}{d \gamma}<0$, as long as $(\phi-1)(\rho-r)-(r+q)+$ $(\phi-1) \frac{\kappa^{2}}{2}<0$. Since $\phi$ solves the quadratic equation $\frac{\kappa^{2}}{2} \phi^{2}+\left(\rho-r-\frac{\kappa^{2}}{2}\right) \phi-(\rho+q)=0$, it follows that $(\phi-1)(\rho-r)-(r+q)+(\phi-1) \frac{\kappa^{2}}{2}=-(\phi-1)^{2} \frac{\kappa^{2}}{2}<0$.

Proof of Proposition 4. The proof of this proposition proceeds in steps. The first two Lemmas establish that the proposed transfer policy will make it possible for an agent who follows the optimal consumption process of proposition 4 to satisfy the intertemporal budget constraint. The proof then continues to show that the wealth process associated with the optimal consumption process of proposition 4, along with the portfolio process (36), will lead to non-negative levels of wealth at all times. Finally, it is shown that the consumption policy of proposition 4, along with the portfolio choice (36), are optimal for an agent who is faced with transfers given by (34) and attain the upper bound of proposition 2 .

Lemma 10 Let $K$ and $\phi$ be given by (28) and (27) and for any $0<\lambda<\xi^{-\gamma}$ let $Z_{t}=\lambda e^{\rho s} H_{s} X_{s}^{*}$. Then

$$
\begin{equation*}
\int_{0}^{\infty} E_{t}\left(\int_{t}^{\infty} e^{-q(s-t)} H_{s} X_{s}^{*} d G_{s}-\int_{t}^{\infty} e^{-q(s-t)} H_{s} X_{s}^{*} Z_{s}^{-\frac{1}{\gamma}} d s\right) d X_{t}^{*}=0 \tag{110}
\end{equation*}
$$

Proof of Lemma 10. It will simplify notation to let

$$
\begin{equation*}
\eta \equiv-K \xi\left(\phi-1+\frac{1}{\gamma}\right) . \tag{111}
\end{equation*}
$$

The first step is to compute

$$
\begin{equation*}
\frac{E_{t} \int_{t}^{\infty} e^{-q s} H_{s} X_{s}^{*} d G_{s}}{e^{-q t} H_{t} X_{t}^{*}}=\eta \frac{E_{t} \int_{t}^{\infty} e^{-q s} H_{s} d X_{s}^{*}}{e^{-q t} H_{t} X_{t}^{*}} \tag{112}
\end{equation*}
$$

Applying integration by parts and using the definition of $Z_{t}$ gives

$$
\begin{equation*}
E_{t}\left(\int_{t}^{\infty} e^{-q s} H_{s} d X_{s}^{*}\right)=\frac{1}{\lambda}\left[-e^{-(\rho+q) t} Z_{t}+E_{t}\left(\int_{t}^{\infty}(r+q) e^{-(\rho+q) s} Z_{s} d s\right)\right] . \tag{113}
\end{equation*}
$$

Using (113) in equation (112) gives

$$
\begin{equation*}
\frac{E_{t} \int_{t}^{\infty} e^{-q s} H_{s} X_{s}^{*} d G_{s}}{e^{-q t} H_{t} X_{t}^{*}}=\eta\left[(r+q) \frac{E_{t}\left(\int_{t}^{\infty} e^{-(\rho+q)(s-t)} Z_{s} d s\right)}{Z_{t}}-1\right] . \tag{114}
\end{equation*}
$$

By using a logic similar to equations (92)-(94),

$$
\begin{equation*}
E_{t}\left(\int_{t}^{\infty} e^{-(\rho+q)(s-t)} Z_{s} d s\right)=-\frac{1}{\phi} \frac{\xi^{-\gamma}}{r+q}\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi}+\frac{1}{r+q} Z_{t} \tag{115}
\end{equation*}
$$

where $\phi$ is defined in equation (27). Plugging back (115) into (114) gives

$$
\begin{equation*}
\frac{E_{t} \int_{t}^{\infty} e^{-q s} H_{s} X_{s}^{*} d G_{s}}{e^{-q t} H_{t} X_{t}^{*}}=-\frac{\eta}{\phi}\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi-1} \tag{116}
\end{equation*}
$$

To conclude the proof, note that equations (90) and (95) imply that

$$
\begin{equation*}
\frac{E_{t}\left(\int_{t}^{\infty} e^{-q s} H_{s} X_{s}^{*} Z_{s}^{-\frac{1}{\gamma}} d s\right)}{e^{-q t} H_{t} X_{t}^{*}}=\frac{E_{t}\left(\int_{t}^{\infty} e^{-(\rho+q)(s-t)} Z_{s}^{1-\frac{1}{\gamma}} d s\right)}{Z_{t}}=\frac{\frac{1}{\phi} \frac{1-\gamma}{\gamma} K \xi^{1-\gamma}\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi}+K Z_{t}^{1-\frac{1}{\gamma}}}{Z_{t}} \tag{117}
\end{equation*}
$$

Combining (117) with (116) gives:

$$
\begin{aligned}
& \frac{E_{t}\left(\int_{t}^{\infty} e^{-q s} H_{s} X_{s}^{*} d G_{s}-\int_{t}^{\infty} e^{-q s} H_{s} X_{s}^{*} Z_{s}^{-\frac{1}{\gamma}} d s\right)}{e^{-q t} H_{t} X_{t}^{*}}= \\
& =-\frac{\eta}{\phi}\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi-1}-\frac{\frac{1}{\phi} \frac{1-\gamma}{\gamma} K \xi^{1-\gamma}\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi}+K Z_{t}^{1-\frac{1}{\gamma}}}{Z_{t}} .
\end{aligned}
$$

Since $d X_{t}^{*} \neq 0$ when and only when $Z_{t}=\xi^{-\gamma}$, equation (110) amounts to checking that:

$$
-\frac{\eta}{\phi}-\left(\frac{1}{\phi} \frac{1-\gamma}{\gamma}+1\right) K \xi=0
$$

which follows easily from the definition of $\eta$.
Lemma 11 Let $Z_{s}$ be as in the statement of the proposition 4 and let $G_{t}$ be as in (34). Then the consumption policy:

$$
\begin{equation*}
c_{s}^{*}=\left(Z_{s}\right)^{-\frac{1}{\gamma}} \tag{118}
\end{equation*}
$$

satisfies:

$$
\begin{equation*}
E \int_{0}^{\infty} e^{-q s} H_{s} X_{s}^{*} c_{s}^{*} d s=W_{0}+\int_{0}^{\infty} e^{-q s} H_{s}\left(X_{s}^{*}-1\right) d G_{s} \tag{119}
\end{equation*}
$$

Proof of Lemma 11. Taking any $\lambda \in\left(0, \xi^{-\gamma}\right]$, using the definition of $X_{t}^{*}$, and equation (110), the same reasoning behind (65) leads to

$$
\begin{align*}
& E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \max _{c_{s}}\left(\frac{c_{s}^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s} H_{s} X_{s}^{*} c_{s}\right) d s+\lambda \int_{0}^{\infty} e^{-q s} H_{s}\left(X_{s}^{*}-1\right) d G_{s}\right)+\lambda W_{0}=(120)  \tag{120}\\
& =E\left[\int_{0}^{\infty} e^{-(\rho+q) s} \frac{\gamma}{1-\gamma}\left(e^{\rho s} \lambda H_{s} X_{s}^{*}\right)^{\frac{\gamma-1}{\gamma}} d s+\int_{0}^{\infty} e^{-(\rho+q) s}\left(e^{\rho s} \lambda H_{s} X_{s}^{*}\right)^{1-\frac{1}{\gamma}}\left(1-\frac{1}{X_{s}^{*}}\right) d s\right]+\lambda W_{0} \tag{121}
\end{align*}
$$

Hence the $\lambda^{*}$ that minimizes (32) (and hence minimizes [121]) also minimizes (120). But since $\lambda$ minimizes (120), the same argument as in He and Pages (1993) (Proof of Theorem 1) leads to (119).

Lemma 11 has asserted that the consumption policy (118) satisfies the intertemporal budget constraint (119). It remains to show that this consumption policy along with the portfolio policy (36) will lead to a process for financial wealth that satisfies $W_{t} \geq 0$. To that end let $\eta$ be given as in (111) and define:

$$
\begin{equation*}
W^{*}\left(Z_{t}\right)=-K\left(\xi^{-\gamma}\right)^{-\frac{1}{\gamma}}\left(\frac{Z_{t}}{\xi^{-\gamma}}\right)^{\phi-1}+K Z_{t}^{-\frac{1}{\gamma}} \tag{122}
\end{equation*}
$$

It is straightforward to verify the following facts about $W^{*}\left(Z_{t}\right)$ :

$$
\begin{align*}
& \frac{\kappa^{2}}{2} Z^{2} W_{Z Z}^{*}+\left(\rho-r+\kappa^{2}\right) Z W_{Z}^{*}-(r+q) W+(Z)^{-\frac{1}{\gamma}}=0  \tag{123}\\
& W^{*}\left(\xi^{-\gamma}\right)=0, W^{*}(Z) \geq 0 \text { for all } Z \in\left(0, \xi^{-\gamma}\right]  \tag{124}\\
& W_{Z}^{*}\left(\xi^{-\gamma}\right)=-K \xi\left(\phi-1+\frac{1}{\gamma}\right)\left(\xi^{-\gamma}\right)^{-1}=\frac{\eta}{\xi^{-\gamma}} \tag{125}
\end{align*}
$$

The next step is to verify that $W^{*}\left(Z_{t}\right)$ is the stochastic process for the financial wealth of the agent. To see this, use the definition of $c_{s}^{*}$ (equation [118]) along with the definitions of $d G_{t}, W_{t}^{*}$ (equations [34] and [122] respectively) and apply Ito's Lemma to obtain:

$$
d\left(\int_{0}^{t} c_{s}^{*} d s-\int_{0}^{t} d G_{s}+W_{t}^{*}\right)=
$$

$$
\begin{aligned}
& =c_{t}^{*} d t-\eta \frac{d X_{t}^{*}}{X_{t}^{*}}+W_{Z}^{*} d Z_{t}+\frac{\kappa^{2}}{2} W_{Z Z}^{*} Z_{t}^{2} d t \\
& =\left(c_{t}^{*}-Z_{t}^{-\frac{1}{\gamma}}\right) d t+\left[W_{Z}^{*}\left(\xi^{-\gamma}\right) \xi^{-\gamma}-\eta\right] \frac{d X_{t}^{*}}{X_{t}^{*}}+(r+q) W_{t}^{*} d t-\kappa^{2} Z_{t} W_{Z}^{*} d t-\kappa W_{Z}^{*} Z_{t} d B_{t}= \\
& =(r+q) W_{t}^{*} d t-\kappa^{2} Z_{t} W_{Z}^{*} d t-\frac{\kappa}{\sigma} W_{Z}^{*} Z_{t}\left(\frac{d P_{t}}{P_{t}}-\mu d t\right) \\
& =(r+q) W_{t}^{*} d t-\kappa^{2} Z_{t} W_{Z}^{*} d t-\frac{\kappa}{\sigma} W_{Z}^{*} Z_{t}\left(\frac{d P_{t}}{P_{t}}-(\mu-r) d t-r d t\right)= \\
& =q W_{t}^{*} d t+r\left(W_{t}^{*}+\frac{\kappa}{\sigma} W_{Z}^{*} Z_{t}\right) d t-\frac{\kappa}{\sigma} W_{Z}^{*} Z_{t} \frac{d P_{t}}{P_{t}}= \\
& =q W_{t}^{*} d t+r\left(W_{t}^{*}-\pi_{t}^{*}\right) d t+\pi_{t}^{*} \frac{d P_{t}}{P_{t}} .
\end{aligned}
$$

Integrating gives

$$
\int_{0}^{t} c_{s}^{*} d s+W_{t}^{*}=W_{0}-D_{0}+\int_{0}^{t} d G_{s}+\int_{0}^{t} q W_{s}^{*} d t+\int_{0}^{t} r\left(W_{t}^{*}-\pi_{t}^{*}\right) d t+\int_{0}^{t} \pi_{t}^{*} \frac{d P_{t}}{P_{t}} .
$$

Hence the process $W_{t}^{*}$ satisfies the equation (12) for an agent who chooses a consumption policy given by (118) and a portfolio policy given by (36). Accordingly, it is the financial wealth process that is associated with that policy pair. Moreover, by equation (124) the financial wealth process is non-negative. Accordingly, the policies given by (118) and (36) are feasible for an agent who is faced with the transfer process (34).

Verifying the optimality of the stated policy pair is simple. According to proposition 1
$V\left(W_{0}\right)=\min _{\lambda>0, X_{s} \in \mathcal{D}}\left[\begin{array}{c}E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \max _{c_{s}}\left(\frac{c_{s}^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s} H_{s} X_{s} c_{s}\right) d s+\lambda \int_{0}^{\infty} e^{-q s} H_{s} X_{s} d G_{s}\right) \\ +\lambda\left(W_{0}-D_{0}\right)\end{array}\right] \leq Q\left(W_{0}\right)$,
where

$$
Q\left(W_{0}\right) \equiv \min _{\lambda>0}\left[\begin{array}{c}
E\left(\int_{0}^{\infty} e^{-(\rho+q) s} \max _{c_{s}}\left(\frac{c_{s}^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s} H_{s} X_{s}^{*} c_{s}\right) d s+\lambda \int_{0}^{\infty} e^{-q s} H_{s} X_{s}^{*} d G_{s}\right) \\
+\lambda\left(W_{0}-D_{0}\right)
\end{array}\right] .
$$

One can use now Lemma 11 to illustrate that the consumption policy (118) leads to a payoff for the agent equal to $Q\left(W_{0}\right)$ which is an upper bound to the value function of the agent $V\left(W_{0}\right)$. Since the consumption policy (118) is also feasible, the payoff associated with that policy also provides a lower bound to the value function $V\left(W_{0}\right)$. Hence this policy must be optimal. Finally, the easiest way to show that

$$
D_{0}=K \xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\left(\frac{\lambda^{*}}{\xi^{-\gamma}}\right)^{\phi-1}
$$

is to observe that the intertemporal budget constraint implies that

$$
E_{\tau_{0}}\left(\int_{\tau_{0}}^{\infty} e^{-q\left(s-\tau_{0}\right)} \frac{H_{s}}{H_{\tau_{0}}} c_{s}^{*} d s\right)=E_{\tau_{0}}\left(\int_{\tau_{0}}^{\infty} e^{-q\left(s-\tau_{0}\right)} \frac{H_{s}}{H_{\tau_{0}}} d G_{s}\right),
$$

where $\tau_{0}$ is the first time that $X_{\tau_{0}} \geq 1$ (or equivalently the first time that $W_{\tau_{0}}=0$ and $\lambda^{*} e^{\rho \tau_{0}} H_{\tau_{0}}=$ $\xi^{-\gamma}$ ). A few manipulations can be used to show that

$$
E_{\tau_{0}}\left(\int_{\tau_{0}}^{\infty} e^{-q\left(s-\tau_{0}\right)} \frac{H_{s}}{H_{\tau_{0}}} c_{s}^{*} d s\right)=\frac{N\left(\xi^{-\gamma}, 1\right)}{\xi^{-\gamma}}=K \xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}
$$

where $N$ is defined and computed in (97) and (96). Finally, since there are no transfers between 0 and $\tau_{0}$ :

$$
\begin{aligned}
D_{0} & =E\left(e^{-q \tau_{0}} H_{\tau_{0}}\right) K \xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}=\frac{1}{\lambda^{*}} E\left(e^{-(\rho+q) \tau_{0}} \lambda^{*} e^{\rho \tau_{0}} H_{\tau_{0}}\right) K \xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}= \\
& =\frac{\xi^{-\gamma}}{\lambda^{*}} E\left(e^{-(\rho+q) \tau_{0}}\right) K \xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}=\left(\frac{\lambda^{*}}{\xi^{-\gamma}}\right)^{\phi-1} K \xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}
\end{aligned}
$$

where the proof of $E\left(e^{-(\rho+q) \tau_{0}}\right)=\left(\frac{\lambda^{*}}{\xi^{-\gamma}}\right)^{\phi}$ is identical to the one given in Oksendal (1998), Chapter 10.

Proof of Proposition 5. Take any transfer process $G_{t}$ such that the resulting consumption process of the agent satisfies $c_{t} \geq \xi$. Proposition 1 implies then that there exists a cumulative multiplier process $X_{t}^{G}$ and a constant $\lambda^{G}$ such that $c_{t}=\left(\lambda^{G} e^{\rho t} H_{t} X_{t}^{G}\right)^{-\frac{1}{\gamma}} \geq \xi$. Letting $X_{t}^{*} \equiv$ $\min \left[1, \frac{\xi^{-\gamma} / \lambda^{G}}{\max _{0 \leq s \leq t}\left(e^{\rho s} H_{s}\right)}\right]$, and $P \equiv E\left(\int_{0}^{\infty} e^{-q s} H_{s} c_{s} d s\right)$ gives

$$
\begin{equation*}
P=E\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda^{G} e^{\rho s} H_{s} X_{s}^{G}\right)^{-\frac{1}{\gamma}} d s\right) \geq E\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda^{G} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right) \tag{126}
\end{equation*}
$$

since ${ }^{38} X_{s}^{*}\left(\lambda^{G}\right) \geq X_{s}^{G}$. Equation (99) implies that

$$
E\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda^{G} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right)=\frac{K \xi^{1-\gamma}}{(\phi-1)} \frac{1}{\gamma}\left(\frac{\lambda^{G}}{\xi^{-\gamma}}\right)^{\phi} \frac{1}{\lambda^{G}}+K\left(\lambda^{G}\right)^{-\frac{1}{\gamma}} .
$$

Combining (101) and (102) implies that the right hand side of the above equation is decreasing in $\lambda^{G}$ whenever $\lambda^{G} \leq \xi^{-\gamma}$. Since $c_{0}=\left(\lambda^{G}\right)^{-\frac{1}{\gamma}} \geq \xi$ this implies furthermore

$$
\begin{align*}
E\left(\int_{0}^{\infty} e^{-q s} H_{s}\left(\lambda^{G} e^{\rho s} H_{s} X_{s}^{*}\right)^{-\frac{1}{\gamma}} d s\right) & \geq \frac{K \xi^{1-\gamma}}{(\phi-1)} \frac{1}{\gamma} \frac{1}{\xi-\gamma}+K \xi=K \xi\left(1+\frac{1}{\phi-1} \frac{1}{\gamma}\right)  \tag{127}\\
& =K \xi\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right) .
\end{align*}
$$

[^23]Combining (126) and (127) concludes the proof.
Proof of Lemma 3. Take any feasible choice of $S_{t}, L_{0}$ that satisfies (43), (44), and (46) and fix the associated processes for $c_{t}, W_{0}$. Then that combination of $c_{t}, W_{0}$ is a feasible choice for the consumer who solves problem 4.

To see this, note first that $W_{0} \geq W^{\text {min }}$ by (43).
Furthermore, the dynamic completeness of markets implies that any combination of $c_{t}, W_{0}$ is feasible for problem 4 as long as it satisfies the requirements ${ }^{39}$

$$
\begin{equation*}
E_{t^{b}} \int_{t^{b}}^{0} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) c_{t} d t+E_{t^{b}} e^{-q \tau}\left(\frac{H_{0}}{H_{t^{b}}}\right) W_{0} \leq E_{t^{b}} \int_{t^{b}}^{0} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) Y d t, \tag{128}
\end{equation*}
$$

and

$$
\begin{equation*}
W_{t}=E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right)\left(c_{u}-Y\right) d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W_{0} \geq 0 \text { for all } t \in\left[t^{b}, 0\right] . \tag{129}
\end{equation*}
$$

To show that the combination of $c_{t}, W_{0}$ that is associated with problem 3 satisfies (128), use (51) and Ito's Lemma to compute $d\left(e^{-q t} H_{t} W_{t}\right)$, integrate and use the fact that $W_{t^{b}}=0, W_{t} \geq 0$ to obtain

$$
\begin{align*}
& E_{t_{b}} \int_{t^{b}}^{0} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) c_{t} d t+E_{t^{b}} e^{-q \tau}\left(\frac{H_{0}}{H_{t^{b}}}\right) W_{0^{-}}=  \tag{130}\\
& E_{t^{b}} \int_{t^{b}}^{0} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) Y d t-E_{t^{b}} \int_{t^{b}}^{0} e^{-q\left(t-t^{b}\right)}\left(\frac{H_{t}}{H_{t^{b}}}\right) d S_{t} .
\end{align*}
$$

Using $W_{0}=W_{0^{-}}+L_{0}$ and (44) inside (130) implies (128).
Finally, it remains to show (129). Since the processes $c_{t}, W_{t}$ associated with $S_{t}, L_{0}$ satisfy $W_{t} \geq 0$, it follows that

$$
\begin{equation*}
E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right)\left(c_{u}-Y\right) d u+E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) d S_{u}+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W_{0^{-}} \geq 0 \tag{131}
\end{equation*}
$$

Adding $E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) L_{0}$ to both sides of the inequality (131) and re-arranging gives

$$
\begin{aligned}
& E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right)\left(c_{u}-Y\right) d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W_{0} \\
& \geq E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) L_{0}-E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) d S_{u} \geq 0,
\end{aligned}
$$

where the last inequality follows from (46). Since any attainable combination of $c_{t}, W_{0}$ for problem 3 is feasible for 4 , this implies that the value function in problem 4 must be at least as high as the respective value function of problem 3.

Proof of Lemma 4 . If $c_{t}^{*}, W_{t}^{*}$ are optimal consumption and wealth processes that solve

[^24]problem 4, then they are linked by the present value relation
\[

$$
\begin{equation*}
W_{t}^{*}=E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right)\left(c_{u}^{*}-Y\right) d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W_{0}^{*} \tag{132}
\end{equation*}
$$

\]

By construction of $\chi$, it follows that

$$
\begin{align*}
E_{\chi} \int_{\chi}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{\chi}}\right) Y d u & =\frac{1-e^{-(r+q)(0-\chi)}}{r+q} Y=e^{-(r+q)(0-\chi)} W^{\min }  \tag{133}\\
& =E_{t} e^{-q(0-\chi)}\left(\frac{H_{0}}{H_{\chi}}\right) W^{\min }
\end{align*}
$$

Adding and subtracting $e^{-(r+q)(0-t)} W^{\text {min }}$ on the right hand side of (132) implies that for any $t>\chi$

$$
\begin{align*}
W_{t}^{*} & =E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) c_{u}^{*} d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right)\left(W_{0}^{*}-W^{\min }\right)  \tag{134}\\
& +e^{-(r+q)(0-t)} W^{\min }-\frac{1-e^{-(r+q)(0-t)}}{r+q} Y
\end{align*}
$$

The first two terms on the right hand side of equation (134) are non-negative (since $W_{0}^{*}-W^{\min } \geq 0$ ), while the sum of the last two terms on the right hand side of equation (134) is positive ${ }^{40}$.

Proof of Proposition 6. The fact that the proposed policy satisfies (43) follows from $W_{0^{-}} \geq 0$. The requirement (44) follows by the construction of $\chi$, while the requirement (46) follows from the fact that $d S_{t}$ is deterministic (see remark in the text).

To conclude the proof, it suffices to show that $V_{t^{b}} \geq J_{t^{b}}$. To that end, let $c_{t}^{*}, W_{0}^{*}$ denote the optimal consumption and wealth processes that solve problem 4. Consider an agent faced with the policy pair $<d S_{t}, L_{0}>$ in the statement of the proposition. To show $V_{t^{b}} \geq J_{t^{b}}$, it suffices to show that $c_{t}^{*}, W_{0}^{*}$ remain feasible choices for this agent.

To that end, note that the wealth process $W_{t}$ for an agent who chooses $c_{t}=c_{t}^{*}$ and $W_{0^{-}}=$ $W_{0}^{*}-L_{0}$ in the presence of the policy pair $<d S_{t}, L_{0}>$ given in the proposition, is given by

$$
W_{t}=E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right)\left(c_{u}^{*}-Y \times 1_{\{t<\chi\}}\right) d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W_{0^{-}},
$$

where $1_{\{t<\chi\}}$ is an indicator function that takes the value 1 if $t<\chi$ and zero otherwise. Clearly,

$$
\begin{aligned}
& { }^{40} \text { Note that } \\
& e^{-(r+q)(0-t)} W^{\min }-\frac{1-e^{-(r+q)(0-t)}}{r+q} Y= \\
& e^{(r+q)(t-\chi)}\left(e^{-(r+q)(0-\chi)} W^{\min }-\frac{e^{(r+q)(\chi-t)}-e^{-(r+q)(0-\chi)}}{r+q} Y\right)>0,
\end{aligned}
$$

where the last inequality follows by equation (133) and $t>\chi$.
$W_{0} \geq 0$ for all $t \geq \chi$, since $c_{u}^{*} \geq 0$ and $W_{0^{-}} \geq 0$. For $t<\chi$, observe that

$$
\begin{align*}
W_{t} & =E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) c_{u}^{*} d u-E_{t} \int_{t}^{\chi} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) Y d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W_{0^{-}}  \tag{135}\\
& =E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right)\left(c_{u}^{*}-Y\right) d u+E_{t} \int_{\chi}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) Y d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W_{0^{-}} \\
& =E_{t} \int_{t}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right)\left(c_{u}^{*}-Y\right) d u+E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right)\left(W_{0^{-}}+W^{\min }\right) \geq 0,
\end{align*}
$$

where the last line in (135) follows from $E_{t} \int_{\chi}^{0} e^{-q(u-t)}\left(\frac{H_{u}}{H_{t}}\right) Y d u=E_{t} e^{-q(0-t)}\left(\frac{H_{0}}{H_{t}}\right) W^{\text {min }}$, the definition $W_{0^{-}}+W^{\min }=W_{0}$, and the fact that the wealth process in problem 4 is non-negative. Clearly, $W_{t}=0$ at $t=t^{b}$, so that the pair $c_{t}=c_{t}^{*}$ and $W_{0^{-}}=W_{0}^{*}-L_{0}$ also satisfies the intertemporal budget constraint at $t=t^{b}$. This verifies that the pair $c_{t}=c_{t}^{*}$ and $W_{0^{-}}=W_{0}^{*}-L_{0}$ is a feasible pair for the agent solving the problem (50) - (52), which implies $V_{t^{b}} \geq J_{t^{b}}$. Combining this with Lemma 3 concludes the proof.

Proof of Lemma 5. Writing out the Bellman equation for an agent and proceeding as in Kobila (1993) leads to the optimality condition $\int_{0}^{\infty}\left(V_{W}^{\mathcal{N I}}-\xi^{-\gamma}\right) d N_{t}^{\mathcal{N I}}=0$. Combining this optimality condition with the first order condition for consumption $V_{W}^{\mathcal{N} \mathcal{I}}=\left(c^{\mathcal{N I} \mathcal{I}}\right)^{-\gamma}$ yields the result.

Proof of Proposition 7. Since the idiosyncratic shock is bounded, and an agent who has experienced an idiosyncratic shock can always set $N_{0^{+}}-N_{0}=\frac{\max \left|\overline{Y_{t}}\right|}{r+q}$, the value function of an agent who has experienced an idiosyncratic shock is bounded. Therefore, as $\theta \rightarrow 0$, the objective in equation (56) converges to the objective of problem 1 (taking into account equation [55]). Also, in light of Lemma 5, any consumption plan that is feasible (for a consumer that has not experienced an idiosyncratic shock) under problem 5 is feasible under problem 1 and vice versa. By the theorem of the maximum the value function of problem 5 converges to the value function of problem 1 as $\theta \rightarrow 0$.

## References

Abel, A. B. (1987). The Ricardian Equivalence Theorem, Volume 4 of In: John Eatwell, Murray Milgate and Peter Newman (eds.), The New Palgrave: A Dictionary of Economic Theory and Doctrine, pp. 174-179. The Macmillan Press Ltd.

Amador, M., I. Werning, and G.-M. Angeletos (2006). Commitment vs. flexibility. Econometrica $74(2)$, 365-396.

Ball, L. and N. G. Mankiw (2007). Intergenerational risk sharing in the spirit of arrow, debreu, and rawls, with applications to social security design. Journal of Political Economy 115(4), 523-547.

Barro, R. J. (1974). Are government bonds net wealth? Journal of Political Economy 82(6), 1095 - 1117.
Barro, R. J. (1979). On the determination of the public debt. Journal of Political Economy 87(5), 940 971.

Basak, S. (2002). A comparative study of portfolio insurance. Journal of Economic Dynamics and Control 26, 1217-1241.

Basak, S. and D. Cuoco (1998). An equilibrium model with restricted stock market participation. Review of Financial Studies 11 (2), 309 - 341.

Blanchard, O. J. (1985). Debt, deficits, and finite horizons. Journal of Political Economy 93(2), 223-247.
Bovbjerg, B. D. (2009). Private pensions: Alternative approaches could address retirement risks faced by workers but pose tradeoffs. Technical Report GAO-09-642, United States Government Accountability Office.

Bovenberg, A. and H. Uhlig (2008). Pension systems and the allocation of macroeconomic risk. in L. Reichlin and K.D. West (Eds.), NBER International Seminar on Macroeconomics 2006 (pp. 241-344). Chicago: University of Chicago Press.

Bulow, J. and K. Rogoff (1989). Sovereign debt: Is to forgive to forget? American Economic Review 79(1), 43-50.

Campbell, J. Y. and M. Feldstein (2001). Risk Aspects of Investment-Based Social Security Reform. Cambridge, MA: NBER.

Chiappori, P.-A., I. Macho, P. Rey, and B. Salanie (1994). Repeated moral hazard: The role of memory, commitment, and the access to credit markets. European Economic review 38, 1527-1553.

Chien, Y., H. Cole, and H. Lustig (2007). A multiplier approach to understanding the macro implications of household finance. National Bureau of Economic Research, Inc, NBER Working Papers: 13555.

Cole, H. L. and N. R. Kocherlakota (2001). Efficient allocations with hidden income and hidden storage. Review of Economic Studies 68(3), 523-42.

Constantinides, G. M., J. B. Donaldson, and R. Mehra (2005). Junior must pay: Pricing the implicit put in privatizing social security. Annals of Finance 1, 1-34.

Cox, J. C., J. Ingersoll, Jonathan E., and S. A. Ross (1985). An intertemporal general equilibrium model of asset prices. Econometrica 53(2), 363-384.

Cuoco, D. (1997). Optimal consumption and equilibrium prices with portfolio constraints and stochastic income. Journal of Economic Theory 72(1), 33-73.

Cvitanic, J. and I. Karatzas (1992). Convex duality in constrained portfolio optimization. Annals of Applied Probability 2, 767-818.

Detemple, J. and A. Serrat (2003). Dynamic equilibrium with liquidity constraints. Review of Financial Studies 16(2), 597-629.

Duffie, D. (2001). Dynamic asset pricing theory. Princeton and Oxford: Princeton University Press.
Dumas, B. and A. Lyasoff (2010). Incomplete-market equilibria solved recursively on an event tree. Working Paper, INSEAD and Boston University.

Farhi, E. and S. Panageas (2007). Saving and investing for early retirement: A theoretical analysis. Journal of Financial Economics 83(1), 87 - 121.

Feldstein, M. (2005a). Reducing the risk of investment-based social security reform. National Bureau of Economic Research, Working Paper: 11084.

Feldstein, M. (2005b). Rethinking social insurance. American Economic Review 95(1), 1-24.
Feldstein, M. and E. Ranguelova (2001). Individual risk in an investment-based social security system. American Economic Review 91 (4), 1116-1125.

Fuster, L., A. Imrohoroglu, and S. Imrohoroglu (2008). Personal Security Accounts and Mandatory Annuitization in a Dynastic Framework. In: Fenge, R., de Menil, G. and Pestieau, P., eds., Strategies for Pension Reform. Cambridge: MIT Press.

Gallmeyer, M. and B. Hollifield (2008). An examination of heterogeneous beliefs with a short-sale constraint in a dynamic economy. Review of Finance 12(2), 323-364.

Golosov, M. and A. Tsyvinski (2007). Optimal taxation with endogenous insurance markets. Quarterly Journal of Economics 122(2), 487-534.

Golosov, M., A. Tsyvinski, and I. Werning (2007). New dynamic public finance: A user's guide. NBER Macroeconomics Annual 2006, Cambridge and London: MIT Press, 317 - 363.

Grossman, S. J. and Z. Zhou (1996). Equilibrium analysis of portfolio insurance. Journal of Finance 51(4), 1379-1403.

Haugh, M., L. Kogan, and J. Wang (2004). Evaluating portfolio policies: A duality approach. Operations Research 54, 405-418.

He, H. and H. F. Pages (1993). Labor income, borrowing constraints, and equilibrium asset prices. Economic Theory 3(4), 663-696.

He, H. and N. D. Pearson (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. Journal of Economic Theory 54(2), 259-304.

Karatzas, I. and S. E. Shreve (1991). Brownian motion and stochastic calculus (Second ed.), Volume 113 of Graduate Texts in Mathematics. New York: Springer-Verlag.

Karatzas, I. and S. E. Shreve (1998). Methods of mathematical finance. Applications of Mathematics. New York: Springer-Verlag.

Kobila, T. (1993). A class of solvable stochastic investment problems involving singular controls. Stochastics and Stochastics Reports 43, 29-63.

Kocherlakota, N. R. (1996). Implications of efficient risk sharing without commitment. Review of Economic Studies 63(4), 595-609.

Kogan, L. and R. Uppal (2001). Risk aversion and optimal portfolio policies in partial and general equilibrium economies. mimeo. MIT.

Krueger, D. and F. Kubler (2006). Pareto-improving social security reform when financial markets are incomplete!? American Economic Review 96(3), 737-755.

Ljungqvist, L. and T. J. Sargent (2004). Recursive macroeconomic theory. Cambridge and London: MIT Press.

Lucas, R. E. and N. L. Stokey (1983). Optimal fiscal and monetary policy in an economy without capital. Journal of Monetary Economics 12(1), 55-93.

Lustig, H. (2002). Essays on risk sharing in economies with limited commitment. PhD Thesis, Stanford.
Marcet, A. and R. Marimon (1998). Recursive contracts. UPF Working Paper.
Mitchell, O. S. and M.-E. Lachance (2003). Guaranteeing individual accounts. American Economic Review $93(2), 257-260$.

Oksendal, B. (1998). Stochastic differential equations: An introduction with applications. Universitext. Berlin: Springer-Verlag.

Panageas, S. (2010). Optimal taxation in the presence of bailouts. Journal of Monetary Economics 57(1), $101-116$.

Shoffner, D., A. G. Biggs, and P. Jacobs (2005). Poverty-level annuitization requirements in social security proposals incorporating personal retirement accounts. U.S. Social Security Administration Office of Policy.

Smetters, K. (2001). The Effect of Pay-When-Needed Benefit Guarantees on the Impact of Social Security Privatization, pp. 91-111. In: John Y. Campbell and Martin Feldstein, (Eds.) Risk Aspects of InvestmentBased Social Security Reform. Cambridge, MA: NBER.

Stiglitz, J. E. (1992). Prices and Queues as Screening Devices in Competitive Markets., pp. 128 - 166. In: Partha Dasgupta, Douglas Gale, Oliver Hart and Eric Maskin (Eds.) Economic analysis of markets and games: Essays in honor of Frank Hahn. MIT Press: Cambridge and London.

Storesletten, K., C. Telmer, and A. Yaron (1999). The risk sharing implications of alternative social security arrangements. Carnegie-Rochester Conference Series on Public Policy 50, 213-59.

Werning, I. (2002). Optimal dynamic taxation and social insurance. mimeo. University of Chicago.


[^0]:    *Contact: spanagea@chicagobooth.edu. I would like to thank Andrew Abel, Emmanuel Farhi, Michael Gallmeyer, Patrick Kehoe, Naryana Kocherlakota, Ali Lazrak, Debbie Lucas, Ellen McGrattan, Nicholas Souleles and participants of seminars and conferences at Boston University, University of British Columbia, Columbia University, Carnegie Mellon University, University of Chicago, Harvard University, University of Lausanne, London School of Economics, Minneapolis FED, Minnesota Macro-Finance conference, MIT, NBER AP Summer Institute, Philadelphia FED, SED, Texas A\&M, and the Wharton Finance faculty lunch for helpful comments and discussions. This paper has previously circulated under the title "Optimal retirement benefit guarantees".

[^1]:    ${ }^{1}$ As is shown shortly, this is a direct consequence of the principle of Ricardian Equivalence.
    ${ }^{2}$ Shoffner et al. (2005) note in a Social Security Report that "...Such individuals might then qualify for, and as a result place a greater burden on, means-tested antipoverty programs."
    ${ }^{3}$ For instance, a recent Government Accountability Office report Bovbjerg (2009) investigates such regulations in the UK, Switzerland and the Netherlands and documents that these countries use some combination of the above measures.
    ${ }^{4}$ See e.g. "The Obama administration is weighing how the government can encourage workers to turn their savings into guaranteed income streams following a collapse in retiree accounts when the stock market plunged." by Theo Francis in http://www.bloomberg.com/apps/news?pid=20603037\&sid=aHFCE999fWR0

[^2]:    ${ }^{5}$ For some examples see e.g. Feldstein (2005b), Feldstein (2005a), Feldstein and Ranguelova (2001), Fuster et al. (2008), Smetters (2001), Mitchell and Lachance (2003), Constantinides et al. (2005), and the numerous contributions in the special NBER volume edited by Campbell and Feldstein (2001)

[^3]:    ${ }^{6}$ We discuss this issue in detail in the appendix.
    ${ }^{7}$ This literature is too voluminous to summarize here. An indicative sample of alternative views on these important issues are contained in , e.g., Storesletten et al. (1999), Krueger and Kubler (2006), and Ball and Mankiw (2007). Related to this paper, Bovenberg and Uhlig (2008) discuss the joint implications of full/paygo funding as well as defined benefit/contribution systems from the perspective of allocating macroeconomic risk. However, they do not discuss the optimal design of a fully funded system, which is the topic of this paper.
    ${ }^{8}$ A practical implication of the full-financing constraint is that all the policies considered in the paper can be implemented by having the government simply specify the properties of optimal claims and mandating that agents purchase the respective financial products by the private sector.
    ${ }^{9}$ For instance, if agents in the real world can divert their assets to other countries, or put them to uses that are beyond the scope of any given imperfect regulation, effectively they can "hide" their true assets and transactions from the government.
    ${ }^{10}$ In the literature this insight is known as "Ricardian Equivalence". Barro (1974) and Abel (1987) contain a modern treatment of this idea that is originally due to D . Ricardo.

[^4]:    ${ }^{11}$ For an alternative, approximately analytic approach for handling portfolio problems with constraints, see Kogan and Uppal (2001).

[^5]:    ${ }^{12}$ See the overview article of Golosov et al. (2007) and references therein.
    ${ }^{13}$ Hidden savings complicate the solution of dynamic incentive problems considerably. See Chiappori et al. (1994) for an early discussion of these issues. An indicative list of papers in dynamic public finance concerned with the issue of unobserved savings includes Werning (2002), Cole and Kocherlakota (2001), Golosov and Tsyvinski (2007) amongst many others that I do not attempt to summarize here. From a formal perspective, the paper is closest to the literature studying problems of (one-sided) limited commitment. This is especially true of the one-sided commitment version of the model in Kocherlakota (1996). The main friction in that model is the requirement that an agent's continuation value function not fall below the level of autarky. The minimum-standard-of-living constraint studied in this paper turns out to have some mathematical similarities to the that requirement. However, in the present framework, an agent's consumption, savings, etc. are all unobserved, and cannot be dictated. This "hidden action / hidden savings" aspect leads to a non-trivial principal-agent problem, which lies at the core of the paper.

[^6]:    ${ }^{14}$ With a few additional technical assumptions the results can be extended to $\gamma<1$, at the cost of lengthier proofs.
    ${ }^{15} F=\left\{F_{t}\right\}$ denotes the $P$-augmentation of the filtration generated by $B_{t}$.

[^7]:    ${ }^{16}$ Extending the model to include a homothetic bequest function is straightforward.

[^8]:    ${ }^{17}$ The idea of handling behavioral problems as principal-agent problems between multiple selves is popular in the literature. For a recent example see, e.g. Amador et al. (2006) and the references therein.

[^9]:    ${ }^{18}$ This latter normalization is without loss of generality since all quantities of interest depend on the ratio of the stochastic discount factor between two points in time, rather than its level.

[^10]:    ${ }^{19}$ To derive this equation, note that in the deterministic case $d\left(e^{-q t} H_{t} W_{t}\right)=e^{-q t} H_{t}\left(d G_{t}-c_{t} d t\right)$. Integrating the left and right hand side of this equation and imposing the requirement $W_{t} \geq 0$ leads to (15).

[^11]:    ${ }^{20}$ Marcet and Marimon (1998) show a similar result in the context of recursive contracts.

[^12]:    ${ }^{24}$ Note that $\log \left(H_{t}\right)-\log \left(H_{0}\right)=-\left(r+0.5 \kappa^{2}\right) t-\kappa\left(B_{t}-B_{0}\right)=-\left(r+0.5 \kappa^{2}\right) t-\frac{\kappa}{\sigma} \sigma\left(B_{t}-B_{0}\right)=-$ $-\left(r+0.5 \kappa^{2}\right) t-\frac{\kappa}{\sigma}\left[\log P_{t}-\log P_{0}-\left(\mu-0.5 \sigma^{2}\right) t\right]=\frac{\kappa}{\sigma}\left(P_{t}-P_{0}\right)+\left(\frac{\kappa}{\sigma}\left(\mu-0.5 \sigma^{2}\right)-\left(r+0.5 \kappa^{2}\right)\right) t$.
    ${ }^{25}$ Section 7 also implies that the behavior of the stochastic discount factor allows the government to infer the agent's initial assets at retirement. However, it is useful to note here that even if the government were not able to infer an agent's assets upon entering retirement, the agent would have every incentive to truthfully report her assets, assuming that the agent can hide, but not overreport her assets. The reason is that the initial payment $D_{0}$ is declining in the amount of available assets upon entering retirement. Hence an agent who would report a lower amount of assets would be making her post-retirement borrowing constraint more binding.

[^13]:    ${ }^{26}$ Recall that $H_{0}=X_{0}^{*}=1$.

[^14]:    ${ }^{27}$ Clearly, for the purposes of this section $t<0$, and $H_{0}$ is a random number, so that it cannot be normalized to 1 as in the post-retirement problem.

[^15]:    ${ }^{28}$ Specifically, Bulow and Rogoff show that whenever $\widetilde{W}_{t}<0$ for any private entity, then there exist a profitable deviation whereby the private entity defaults on its lenders at $t$, keeps receiving its income from $t$ onwards (in the case of a pension fund the contributions $d S_{t}$ ), and finances its consumption (in the case of a pension fund the terminal payout $L_{0}$ ) without ever having to borrow from its lenders in the future.

[^16]:    ${ }^{29}$ It is useful to remark that in the post-retirement problem $\widetilde{W}_{t}=E_{t} \int_{t}^{\infty} e^{-q(u-t)} H_{u} d G_{u} \geq 0$, and the constraint (46) is automatically satisfied.

[^17]:    ${ }^{30}$ Of course in general equilibrium care should be taken to make sure that aggregate consumption stays above the level $\xi$ multiplied by the mass of retirees. In an endowment economy this could be done by boundedness assumptions on the aggregate endowment. Alternatively one could introduce production in the spirit of Cox et al. (1985).

[^18]:    ${ }^{31}$ The idea that queues can act as devices to elicit hidden information is well established in the literature. See e.g. Stiglitz (1992).
    ${ }^{32}$ Indeed, many successful real-world programs to fight poverty and homelessness are associated with providing subsidized work.
    ${ }^{33}$ The conclusions of the paper would not be materially affected if agents paid a once-and-out pecuniary cost to enter the welfare system, as I explain in the remark at the end of this section.

[^19]:    ${ }^{34}$ Because of the distortions associated with labor taxation (and the deadweight costs associated with screening), separation of types is optimal, in the sense that only agents who experience an idiosyncratic shock should be receiving welfare transfers $d N_{t}$.

[^20]:    ${ }^{35}$ For a textbook treatment, see e.g., Ljungqvist and Sargent (2004).

[^21]:    ${ }^{36}$ For the Skorohod equation see Karatzas and Shreve (1991) p. 210.

[^22]:    ${ }^{37}$ To see this distinguish cases. When $X_{s}^{*}=1$, then solving $\frac{\partial \widetilde{A}(s)}{\partial N_{s}}=0$ gives $N_{s}=1 \leq Q_{s}$. Hence $N_{s}$ is the unique interior solution. When $X_{s}^{*}<1$, then $\frac{\partial \widetilde{A}(s)}{\partial N_{s}}>0$ for all $N_{s} \leq Q_{s}=1$. Hence the solution is given by the corner $N_{s}=Q_{s}=1$.

[^23]:    ${ }^{38}$ This is an implication of the Skorohod equation. See Karatzas and Shreve (1991).

[^24]:    ${ }^{39}$ For a proof, see e.g. Karatzas and Shreve (1998), Chapter 3 or He and Pages (1993).

