Volume 108, Issue 3, June 2013
ISSN 0304-405X

JOURNAL OF Financial ECONOMICS

Contents

Published by ELSEVIER
in collaboration with the
WILLIAM E. SIMON GRADUATE SCHOOL
OF BUSINESS ADMINISTRATION,
UNIVERSITY OF ROCHESTER

Available online at www.sciencedirect.com
SciVerse ScienceDirect

Journal of Financial Economics 108 (2013) 608–614

Contents lists available at SciVerse ScienceDirect

Journal of Financial Economics

journal homepage: www.elsevier.com/locate/jfec

General equilibrium with heterogeneous participants and discrete consumption times ☆

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ARTICLE INFO

Article history:
Received 15 November 2011
Received in revised form
14 May 2012
Accepted 18 June 2012
Available online 16 January 2013

JEL classification:
D51
D58
E43
C68

Keywords:
General equilibrium
Computable general equilibrium models
Production economies
Term structure of interest rates
Determination of interest rates

ABSTRACT

The paper investigates the term structure of interest rates imposed by equilibrium in a production economy consisting of participants with heterogeneous preferences. Consumption is restricted to an arbitrary number of discrete times. The paper contains an exact solution to market equilibrium and provides an explicit constructive algorithm for determining the state price density process. The convergence of the algorithm is proven. Interest rates and their behavior are given as a function of economic variables.

1. Introduction

Interest rates are determined by the equilibrium of supply and demand. Increased demand for credit brings interest rates higher, while an increase in demand for fixed income investment causes rates to go down. To determine the mechanism by which economic forces and investors' preferences cause changes in supply and demand, it is necessary to develop a general equilibrium model of the economy. Such model provides a means of quantitative analysis of how economic conditions and scenarios affect interest rates.

Vasicek (2005) investigates an economy in continuous time with production subject to uncertain technological changes described by a state variable. Consumption is assumed to be in continuous time, with each investor maximizing the expected utility from lifetime consumption. The participants have constant relative risk aversion, with different degrees of risk aversion and different time preference functions. After identifying the optimal investment and consumption strategies, the paper derives conditions for equilibrium and provides a description of interest rates.

For a meaningful economic analysis, it is essential that a general equilibrium model allows heterogeneous participants. If all participants have identical preferences, then they will all hold the same portfolio. Since there is no borrowing and lending in the aggregate, there is no net holding of debt securities by any participant, and no investor is exposed to interest rate risk. Moreover, if the utility functions are all the same, it does not allow for study of how interest rates depend on differences in investors' preferences.

The main difficulty in developing a general equilibrium model with heterogeneous participants had been the need to carry the individual wealth levels as state

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0304-405X/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jfineco.2013.01.005

variables, because the equilibrium depends on the distribution of wealth across the participants. This can be avoided if the aggregate consumption can be expressed as a function of a Markov process, in which case only this Markov process becomes a state variable. This is often simple in models of pure exchange economies, where the aggregate consumption is exogenously specified.

The situation is different in models of production economies. In such economies, the aggregate consumption depends on the social welfare function weights. Because these weights are determined endogenously, it is necessary that the individual consumption levels themselves be functions of a Markov process. This has precluded an analysis of equilibrium in a production economy with any meaningful number of participants; most explicit results for production economies had previously been limited to models with one or two participants.

The above approach is exploited here. Vasicek (2005) shows that the individual wealth levels can be represented as functions of a single process, which is jointly Markov with the technology state variable. This allows construction of equilibrium models with just two state variables, regardless of the number of participants in the economy.

In Vasicek (2005), the equilibrium conditions are used to derive a nonlinear partial differential equation whose solution determines the term structure of interest rates. While the solution to the equation can be approximated by numerical methods, the nonlinearity of the equation could present some difficulties.

The present paper provides the exact solution for the case that consumption takes place at a finite number of discrete times. This solution does not require solving partial differential equations, and explicit computational procedure is provided. If the time points are chosen to be dense enough, the discrete case will approximate the continuous case with the desired precision. Some may in fact argue that, in reality, consumption is discrete rather than continuous, and therefore the discrete case addressed here is the more relevant.

The following section of this paper summarizes the relevant results from Vasicek (2005). Section 3 contains the solution for the equilibrium state price density process and the structure of interest rates in the discrete consumption case. Section 4 gives a proof that the proposed algorithm converges to the market equilibrium.

2. The equilibrium economy

Assume that a continuous time economy contains a production process whose rate of return dA/A on investment is

$$\frac{dA}{A} = \mu dt + \sigma dy, \tag{1}$$

where $y(t)$ is a Wiener process. The process $A(t)$ represents a constant return-to-scale production opportunity. An investment of an amount W in the production at time t yields the amount $WA(s)/A(t)$ at time $s>t$. The production process can be viewed as an exogenously given asset that is available for investment in any amount. The amount of investment in production, however, is determined endogenously.

The parameters of the production process can themselves be stochastic. It will be assumed that their behavior is driven by a Markov state variable X, $\mu=\mu(X(t),t)$, $\sigma=\sigma(X(t),t)$. The dynamics of the state variable, which can be interpreted as representing the state of the production technology, is given by

$$dX = \zeta dt + \psi dy + \varphi dx, \tag{2}$$

where $x(t)$ is a Wiener process independent of $y(t)$. The parameters ζ, ψ, and φ are functions of $X(t)$ and t.

It is assumed that investors can issue and buy any derivatives of any of the assets and securities in the economy. The investors can lend and borrow among themselves, either at a floating short rate or by issuing and buying term bonds. The resultant market is complete. It is further assumed that there are no transaction costs and no taxes or other forms of redistribution of social wealth. The investment wealth and asset values are measured in terms of a medium of exchange that cannot be stored unless invested in the production process. For instance, this wealth unit could be a perishable consumption good.

Suppose that the economy has n participants and let $W_k(0)>0$ be the initial wealth of the k–th investor. Each investor maximizes the expected utility from lifetime consumption,

$$\max E\int_0^T p_k(t)U_k(c_k(t))dt, \tag{3}$$

where $c_k(t)$ is the rate of consumption at time t, $U_k(c)$ is a utility function with $U_k'>0$, $U_k''<0$, and $p_k(t)\geq 0$, $0\leq t\leq T$ is a time preference function. Consider specifically the class of isoelastic utility functions, written in the form

$$\begin{aligned} U_k(c) &= \frac{c^{(\gamma_k-1)/\gamma_k}}{\gamma_k-1} \quad \gamma_k>0,\ \gamma_k\neq 1 \\ &= \log c \quad \gamma_k=1. \end{aligned} \tag{4}$$

Here γ_k is the reciprocal of the relative risk aversion coefficient, $1/\gamma_k=-cU_k''/U_k'$, which will be called the risk tolerance.

An economy cannot be in equilibrium if arbitrage opportunities exist in the sense that the returns on an asset strictly dominate the returns on another asset. A necessary and sufficient condition for absence of arbitrage is that there exist processes $\lambda(t)$, $\eta(t)$, called the market prices of risk for the risk sources $y(t)$, $x(t)$, respectively, such that the price P of any asset in the economy satisfies the equation

$$\frac{dP}{P} = (r+\beta\lambda+\delta\eta)dt + \beta dy + \delta dx, \tag{5}$$

where β, δ are the exposures of the asset to the two risk sources. In particular,

$$\mu = r + \sigma\lambda. \tag{6}$$

It is assumed that Novikov's condition holds,

$$E\exp\left(\tfrac{1}{2}\int_0^T(\lambda^2+\eta^2)dt\right)<\infty. \tag{7}$$

Let Z be the numeraire portfolio of Long (1990) with the dynamics

$$\frac{dZ}{Z} = \left(r+\lambda^2+\eta^2\right)dt + \lambda dy + \eta dx, \tag{8}$$

such that the price P of any asset satisfies

$$\frac{P(t)}{Z(t)} = E_t \frac{P(s)}{Z(s)}. \tag{9}$$

Specifically, the price $B(t,s)$ at time t of a default-free bond with unit face value maturing at time s is given by the equation

$$B(t,s) = E_t \frac{Z(t)}{Z(s)}. \tag{10}$$

Here and throughout, the symbol E_t denotes expectation conditional on a filtration $\Im_t$ generated by $y(t)$, $x(t)$. In integral form, the numeraire portfolio can be written as

$$Z(s) = Z(t)\exp\left(\int_t^s r\mathrm{d}\tau + \tfrac{1}{2}\int_t^s (\lambda^2+\eta^2)\mathrm{d}\tau + \int_t^s \lambda \mathrm{d}y + \int_t^s \eta \mathrm{d}x\right). \tag{11}$$

The process $Z(t)$ is the reciprocal of the state price density process.

Vasicek (2005) shows that the optimal consumption rate of the k–th investor is a function of the numeraire process only, given as

$$c_k(t) = v_k p_k^{\gamma_k}(t) Z^{\gamma_k}(t), \tag{12}$$

where

$$v_k = \frac{W_k(0)}{Z(0)E\int_0^T p_k^{\gamma_k}(t)Z^{\gamma_k-1}(t)\mathrm{d}t} \tag{13}$$

is a constant. The individual wealth level W_k under an optimal strategy is

$$W_k(t) = v_k Z(t) E_t \int_t^T p_k^{\gamma_k}(\tau) Z^{\gamma_k-1}(\tau)\mathrm{d}\tau. \tag{14}$$

The behavior of the wealth level W_k is fully determined by the process $Z(t)$. Moreover, the process $(X(t), Z(t))$ is Markov. That means that $W_k(t)=W_k(X(t),Z(t),t)$ is a function of two state variables X and Z only.

In equilibrium, the total wealth

$$W(t) = \sum_{k=1}^{n} W_k(t) \tag{15}$$

must be invested in the production process (which justifies referring to the production process as the market portfolio). Any lending and borrowing (including lending and borrowing implicit in issuing and buying contingent claims) is among the participants in the economy, and its sum must be zero. Thus, the total exposure to the process y is that of the total wealth invested in the production, and the total exposure to the process x is zero. This produces the equation

$$\mathrm{d}W = \mu W \mathrm{d}t + \sigma W \mathrm{d}y - \sum_{k=1}^{n} v_k p_k^{\gamma_k} Z^{\gamma_k}\mathrm{d}t \tag{16}$$

describing the dynamics of the total wealth. The terminal condition is

$$W(T) = 0. \tag{17}$$

The process Z is further subject to the requirement that

$$\frac{A(t)}{Z(t)} = E_t \frac{A(s)}{Z(s)}. \tag{18}$$

The unique solution of the stochastic differential Eq. (16) subject to Eqs. (17) and (18) is given by

$$W(t) = Z(t)E_t \int_t^T \sum_{k=1}^{n} v_k p_k^{\gamma_k}(\tau) Z^{\gamma_k-1}(\tau)\mathrm{d}\tau. \tag{19}$$

In Vasicek (2005), the process $Z(t)$ is determined in the following manner: Write $W(t)=W(X,Z,t)$ as a function of the state variables. Expanding $\mathrm{d}W$ in Eq. (16) by Ito's lemma and comparing the coefficients of $\mathrm{d}t$, $\mathrm{d}y$, and $\mathrm{d}x$ provides equations from which λ, η can be eliminated, resulting in a nonlinear partial differential equation with known coefficients. Once the function $W(X,Z,t)$ has been determined as the unique solution of this equation, λ and η are calculated from $W(X,Z,t)$ as functions of X, Z, and t. The process $Z(t)$ is obtained by integrating the stochastic differential Eq. (8). Bond prices are determined from Eq. (10).

In the case of discrete consumption dealt with in this paper, the partial differential equation and the subsequent integration of Eq. (8) is replaced by an explicit algorithm described in Section 3.

Equilibrium is fully described by specification of the process $Z(t)$, which determines the pricing of all assets in the economy, such as bonds and derivative contracts, by means of Eq. (9). Solving for the equilibrium requires determining the values of the constants v_1, $v_2,\ldots,$ v_n. The algorithm proposed in this paper utilizes the fact that any choice of the constants is consistent with a unique equilibrium described by the process $Z(t)$, except that the corresponding initial wealth levels calculated as

$$W_k'(0) = Z(0)E\int_0^T v_k p_k^{\gamma_k}(t) Z^{\gamma_k-1}(t)\mathrm{d}t \tag{20}$$

do not agree with the given initial values $W_k(0)$. Repeatedly replacing v_k by $v_k W_k(0)/W_k'(0)$ and recalculating Z converges to the required equilibrium, as proven in Section 4. This is analogous to the method proposed by Negishi (1960) in a deterministic economy.

In economic literature, the usual approach to investigating the existence and uniqueness of equilibrium has been the concept of a representative agent (see Negishi, 1960, and Karatzas and Shreve, 1998). The representative agent maximizes an objective (the social welfare function)

$$\max E\int_0^T \max_{c_1+c_2+\ldots+c_n=c} \sum_{k=1}^{n} \Lambda_k p_k(t) U_k(c_k(t))\mathrm{d}t, \tag{21}$$

where $c(t)$ is the consumption rate of the agent (equal to the aggregate consumption of all participants) and $\Lambda_1, \Lambda_2,\ldots, \Lambda_n$ are weights assigned to the individual participants. The constants v_1, $v_2,\ldots,$ v_n in Eq. (12) are related to the representative agent weights. Eq. (4.5.7) in Theorem 4.5.2 of Karatzas and Shreve (1998) can be written as

$$c_k(t) = \gamma_k^{-\gamma_k} \Lambda_k^{\gamma_k} p_k^{\gamma_k}(t) Z^{\gamma_k}(t). \tag{22}$$

Comparing Eqs. (22) and (12) yields the relationship

$$\Lambda_k = \gamma_k v_k^{1/\gamma_k} \tag{23}$$

for $k=1, 2,\ldots, n$.

3. Discrete consumption times

This paper considers an economy in which consumption takes place only at specific discrete dates. The economy exists in continuous time, and between the consumption dates the participants are continuously trading and the production is continuous. The market is assumed to be complete.

Suppose each investor's time preference function is concentrated at positive points $t_1 < t_2 < \ldots < t_m = T$, so that the k-th investor maximizes the expected utility

$$\max E \sum_{i=1}^{m} p_{ik} U_k(C_{ik}), \tag{24}$$

where C_{ik} is the consumption at time t_i and U_k is a utility function given by Eq. (4). It is assumed that

$$\sum_{k=1}^{n} p_{mk} > 0. \tag{25}$$

Let $Y(t) = 1/Z(t)$ be the state price density process. Put

$$A_i = A(t_i),$$
$$X_i = X(t_i),$$
$$Y_i = Y(t_i) \tag{26}$$

for $i = 0, 1, \ldots, m$, with $t_0 = 0$. The state variable $X(t)$ can be a vector. Furthermore, let

$$N_i = \frac{W(t_i+)}{A_i} \tag{27}$$

for $i = 0, 1, \ldots, m-1$, and $N_m = 0$.

The optimal individual consumption is given from Eq. (12) by

$$C_{ik} = v_k p_{ik}^{\gamma_k} Y_i^{-\gamma_k} \tag{28}$$

for $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$, where v_k are positive constants satisfying the equation

$$v_k = \frac{Y_0 W_k(0)}{E \sum_{i=1}^{m} p_{ik}^{\gamma_k} Y_i^{-\gamma_k+1}}. \tag{29}$$

Eq. (16) takes the form

$$W(t) = N_i A(t) \text{ for } t_i \le t < t_{i+1}, \quad i = 0,1,\ldots,m-1 \tag{30}$$

and

$$N_{i-1} - N_i = \frac{K_i(Y_i)}{A_i} \quad i = 1,2,\ldots,m, \tag{31}$$

where

$$K_i(Y) = \sum_{k=1}^{n} v_k p_{ik}^{\gamma_k} Y^{-\gamma_k} \quad i = 1,2,\ldots,m. \tag{32}$$

From Eq. (31),

$$N_0 = \sum_{i=1}^{m} \frac{K_i(Y_i)}{A_i}. \tag{33}$$

From Eq. (18),

$$Y_{i-1} = E_{t_{i-1}} \frac{A_i}{A_{i-1}} Y_i, \quad i = 1,2,\ldots,m. \tag{34}$$

Note that Eqs. (33) and (34) imply

$$W(0) = \frac{1}{Y_0} E \sum_{i=1}^{m} Y_i K_i(Y_i), \tag{35}$$

as is easily established by multiplying Eq. (33) by $A_m Y_m / Y_0$ and taking expectation.

The solution to Eqs. (31) and (34) subject to $N_m = 0$, $N_0 = W(0)/A(0)$ is obtained by successive elimination of $Y_m, Y_{m-1}, \ldots, Y_1$ and $N_{m-1}, N_{m-2}, \ldots, N_1$. Let K_m^{-1} be the inverse of the function K_m and define recursively two sets of functions G, H as follows:

$$G_m(N,A,X) = K_m^{-1}(NA) \tag{36}$$

and $G_i(N, A, X) = Y$ is the positive solution of the equation

$$Y = H_i\left(N - \frac{K_i(Y)}{A}, A, X\right) \tag{37}$$

for $i = 1, 2, \ldots, m-1$; and

$$H_i(N,A,X) = E_{t_i}\left[\frac{A_{i+1}}{A_i} G_{i+1}(N, A_{i+1}, X_{i+1}) \,|\, A_i = A, X_i = X\right] \tag{38}$$

for $i = 0, 1, \ldots, m-1$. Then

$$Y_i = H_i(N_i, A_i, X_i) \quad i = 0,1,\ldots,m-1$$
$$= G_i(N_{i-1}, A_i, X_i) \quad i = 1,2,\ldots,m. \tag{39}$$

The state price density process at time t is

$$Y(t) = E_t \frac{A_i}{A(t)} Y_i \quad \text{for } t_{i-1} \le t \le t_i,\ i = 1,2,\ldots,m. \tag{40}$$

The above represents the exact solution to the equilibrium economy in the case that consumption is limited to a number of discrete times, provided Eq. (29) hold.

Calculation of the equilibrium solution proceeds as follows: Choose initial values of the constants $v_1, v_2, \ldots, v_n$. A reasonable initial guess is

$$v_k = \frac{W_k(0) A_0^{\gamma_k - 1}}{E \sum_{i=1}^{m} p_{ik}^{\gamma_k} A_i^{\gamma_k - 1}} \tag{41}$$

for $k = 1, 2, \ldots, n$. Calculate recursively the functions G_i, $i = 1, 2, \ldots, m$ and H_i, $i = 0, 1, \ldots, m-1$ from Eqs. (36), (37), and (38). Calculate $Y_0 = H_0(N_0, A_0, X_0)$ and determine $Y_1, Y_2, \ldots, Y_m$ from Eqs. (39) and (31). Calculate $W_k'(0)$ as

$$W_k'(0) = \frac{v_k}{Y_0} E \sum_{i=1}^{m} p_{ik}^{\gamma_k} Y_i^{-\gamma_k+1} \tag{42}$$

for $k = 1, 2, \ldots, n$. Set new values of constants $v_1, v_2, \ldots, v_n$ as

$$v_k' = v_k \frac{W_k(0)}{W_k'(0)}. \tag{43}$$

Repeat the above calculations with the new values of the constants until $W_k'(0)$ are sufficiently close to $W_k(0)$, $k = 1, 2, \ldots, n$. The state price density process is given by Eq. (40). Bond prices are given as

$$B(t,s) = E_t \frac{Y(s)}{Y(t)}. \tag{44}$$

Interest rates are determined by bond prices.

In the special case that $\gamma_k = \gamma$, $k = 1, 2, \ldots, n$, the functions take the form $G_i(N,A,X) = (NA)^{-1/\gamma}(F_i(X) + q_i)^{1/\gamma}$,

$i=1, 2,\ldots, m$, $H_i(N,A,X)=(NA)^{-1/\gamma}F_i^{1/\gamma}(X)$, $i=0, 1,\ldots, m-1$, where $F_m(X)=0$,

$$F_i(X)=\left(E_{t_i}\left[\left(\frac{A_{i+1}}{A_i}\right)^{(\gamma-1)/\gamma}(F_{i+1}(X_{i+1})+q_{i+1})^{1/\gamma}|X_i=X\right]\right)^{\gamma}$$

$$i=0,1,\ldots,m-1 \tag{45}$$

and

$$q_i=\sum_{k=1}^{n}v_k p_{ik}^{\gamma}. \tag{46}$$

Then

$$Y_0=N_0^{-1/\gamma}A_0^{-1/\gamma}F_0{}^{1/\gamma}(X_0) \tag{47}$$

and

$$Y_i=N_0^{-1/\gamma}A_i^{-1/\gamma}(F_i(X_i)+q_i)^{1/\gamma}\prod_{j=1}^{i-1}\left(1+\frac{q_j}{F_j(X_j)}\right)^{1/\gamma} i=1,2,\ldots,m. \tag{48}$$

4. Proof of convergence

It will first be shown that $G_i(N,A,X)$, $H_i(N,A,X)$ are decreasing functions of the first argument. Suppose, for some $1\le i\le m$, $G_i(N,A,X)$ is a decreasing function of N. It follows from Eq. (38) that $H_{i-1}(N,A,X)$ is also decreasing in N. Denote by $N=H_i^{-1}(Y,A,X)$ the inverse of the function $Y=H_i(N, A, X)$ with respect to the first argument while keeping the remaining arguments constant. Then, from Eq. (37),

$$H_{i-1}^{-1}(G_{i-1}(N,A,X),A,X)+\frac{K_{i-1}(G_{i-1}(N,A,X))}{A}=N. \tag{49}$$

The expression on the left-hand side of this equation is a decreasing function of G_{i-1} and therefore the function $G_{i-1}(N,A,X)$ is decreasing in N. Because $G_m(N,A,X)$ is decreasing in N, it follows by induction that $G_i(N,A,X)$, $i=1, 2,\ldots, m$, and consequently $H_i(N,A,X)$, $i=0, 1,\ldots, m-1$, are all decreasing functions of the first argument.

From Eq. (39),

$$Y_i=G_i(N_{i-1},A_i,X_i)=G_i(H_{i-1}^{-1}(Y_{i-1},A_{i-1},X_{i-1}),A_i,X_i) \tag{50}$$

for $i=1, 2,\ldots, m$. Define the function Q_m as

$$\begin{aligned}&Q_m(N,A_1,A_2,\ldots,A_m,X_1,X_2,\ldots,X_m)\\&\quad=G_m(H_{m-1}^{-1}(G_{m-1}(\ldots H_1^{-1}(G_1(N,A_1,X_1),A_1,X_1)\ldots,\\&\qquad A_{m-1},X_{m-1}),A_{m-1},X_{m-1}),A_m,X_m).\end{aligned} \tag{51}$$

Since there is an odd number of decreasing functions in the nested expression (51), Q_m is a decreasing function of N. Then

$$Y_m=Q_m(N_0,A_1,A_2,\ldots,A_m,X_1,X_2,\ldots,X_m). \tag{52}$$

Note that (52) represents the solution to Eqs. (33) and (34), since the intermediate values of $N_1, N_2,\ldots, N_{m-1}$ have been eliminated.

Assume that $\gamma_k\ge 1$, $k=1, 2,\ldots, n$ (corresponding to the sufficient condition (4.6.4) for uniqueness of the equilibrium solution in Theorem 4.6.1 in Karatzas and Shreve, 1998). Let $v_1, v_2,\ldots, v_n$ be arbitrary positive constants and determine $Y_0, Y_1,\ldots, Y_m$ from Eq. (39). Calculate $W_k'(0)$ from Eq. (42) and v_k' from Eq. (43), $k=1, 2,\ldots, n$. Put

$$K_i'(Y)=\sum_{k=1}^{n}v_k'p_{ik}^{\gamma_k}Y^{-\gamma_k} \tag{53}$$

and denote by $Y_0',Y_1',\ldots,Y_m'$ the variables calculated using the constants $v_1',v_2',\ldots,v_n'$ in place of $v_1, v_2,\ldots, v_n$. Then

$$N_0=\sum_{i=1}^{m}\frac{K_i'(Y_i')}{A_i} \tag{54}$$

and

$$Y_{i-1}'=E_{t_{i-1}}\frac{A_i}{A_{i-1}}Y_i',\quad i=1,2,\ldots,m. \tag{55}$$

Put

$$W_k''(0)=\frac{v_k'}{Y_0'}E\sum_{i=1}^{m}p_{ik}^{\gamma_k}Y_i'^{-\gamma_k+1} \tag{56}$$

and

$$v_k''=v_k'\frac{W_k(0)}{W_k''(0)}. \tag{57}$$

Define

$$\begin{aligned}a_k&=\frac{v_k}{v_k'}=\frac{W_k'(0)}{W_k(0)},\\a_k'&=\frac{v_k'}{v_k'}=\frac{W_k''(0)}{W_k(0)}.\end{aligned} \tag{58}$$

Then

$$a_k'=\frac{Y_0E\sum_{i=1}^{m}p_{ik}^{\gamma_k}Y_i'^{-\gamma_k+1}}{Y_0'E\sum_{i=1}^{m}p_{ik}^{\gamma_k}Y_i^{-\gamma_k+1}}. \tag{59}$$

Set

$$\begin{aligned}b_k&=a_k^{1/\gamma_k},\\b_k'&=a_k'^{1/\gamma_k},\end{aligned} \tag{60}$$

$k=1, 2,\ldots, n$. Let $b_{\min}$, $b_{\max}$ be the lowest and highest value, respectively, of $b_1, b_2,\ldots, b_n$ and $b_{\min}',b_{\max}'$ be the lowest and highest value, respectively, of $b_1',b_2',\ldots,b_n'$. Put

$$\alpha_k=\frac{W_k(0)}{W(0)}, \tag{61}$$

$k=1, 2,\ldots, n$. Note that

$$\sum_{k=1}^{n}\alpha_k a_k=\sum_{k=1}^{n}\alpha_k a_k'=\sum_{k=1}^{n}\alpha_k=1 \tag{62}$$

and therefore

$$b_{\min}\le 1\le b_{\max}. \tag{63}$$

Define

$$V_i=b_{\min}Y_i' \tag{64}$$

and put

$$M_0=\sum_{i=1}^{m}\frac{K_i(V_i)}{A_i}. \tag{65}$$

The values $V_1, V_2,\ldots, V_m$ satisfy the relationship

$$V_{i-1}=E_{t_{i-1}}\frac{A_i}{A_{i-1}}V_i,\quad i=1,2,\ldots,m. \tag{66}$$

Eqs. (65) and (66) have the solution

$$V_m = Q_m(M_0, A_1, A_2, \ldots, A_m, X_1, X_2, \ldots, X_m). \tag{67}$$

Now

$$K_i'(Y) = \sum_{k=1}^{n} \frac{v_k}{a_k} p_{ik}^{\gamma_k} Y^{-\gamma_k} = \sum_{k=1}^{n} v_k p_{ik}^{\gamma_k} (b_k Y)^{-\gamma_k}$$
$$\le \sum_{k=1}^{n} v_k p_{ik}^{\gamma_k} (b_{\min} Y)^{-\gamma_k} = K_i(b_{\min} Y) \tag{68}$$

for $i=1, 2, \ldots, m$, and consequently

$$N_0 = \sum_{i=1}^{m} \frac{K_i'(Y_i')}{A_i} \le \sum_{i=1}^{m} \frac{K_i(b_{\min} Y_i')}{A_i} = M_0. \tag{69}$$

Because Q_m is a decreasing function of its first argument, Eqs. (52) and (67) imply

$$Y_m \ge V_m = b_{\min} Y_m'. \tag{70}$$

It is proven similarly that

$$Y_m \le b_{\max} Y_m', \tag{71}$$

and from Eqs. (34) and (55) it then follows that

$$b_{\min} Y_i' \le Y_i \le b_{\max} Y_i' \tag{72}$$

for $i=0, 1, \ldots, m$.

From Eq. (59),

$$b_{\min}^{\gamma_k - 1} \frac{Y_0}{Y_0'} \le a_k' \le b_{\max}^{\gamma_k - 1} \frac{Y_0}{Y_0'} \tag{73}$$

and consequently

$$b_{\min}^{1-1/\gamma_k} (Y_0/Y_0')^{1/\gamma_k} \le b_k' \le b_{\max}^{1-1/\gamma_k} (Y_0/Y_0')^{1/\gamma_k}. \tag{74}$$

If $Y_0/Y_0' \le 1$, then

$$b_{\min} \le b_k' \le b_{\max}^{1-1/\gamma_k} \le b_{\max}^{1-1/\gamma_{\max}} \tag{75}$$

and

$$\frac{b_{\max}'}{b_{\min}'} \le \frac{b_{\max}}{b_{\min}} b_{\max}^{-1/\gamma_{\max}}. \tag{76}$$

If $Y_0/Y_0' \ge 1$, then

$$b_{\min}^{1-1/\gamma_{\max}} \le b_{\min}^{1-1/\gamma_k} \le b_k' \le b_{\max} \tag{77}$$

and

$$\frac{b_{\max}'}{b_{\min}'} \le \frac{b_{\max}}{b_{\min}} b_{\min}^{1/\gamma_{\max}}. \tag{78}$$

Thus, either the inequality (76) or (78) holds.

Put $b_{\max}/b_{\min} = s \ge 1$ and let l be such that $b_l = b_{\min}$. Then

$$1 = \sum_{k=1}^{n} \alpha_k b_k^{\gamma_k} = \alpha_l b_{\min}^{\gamma_l} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_k^{\gamma_k}$$
$$= \alpha_l s^{-\gamma_l} b_{\max}^{\gamma_l} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_k^{\gamma_k} \le \alpha_l s^{-\gamma_l} b_{\max}^{\gamma_l} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_{\max}^{\gamma_k}$$
$$\le \alpha_l s^{-\gamma_{\min}} b_{\max}^{\gamma_{\max}} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_{\max}^{\gamma_{\max}}$$
$$= b_{\max}^{\gamma_{\max}} (\alpha_l s^{-\gamma_{\min}} + 1 - \alpha_l) \le b_{\max}^{\gamma_{\max}} (\alpha_{\min} s^{-\gamma_{\min}} + 1 - \alpha_{\min}) \tag{79}$$

and therefore

$$b_{\max} \ge (\alpha_{\min} s^{-\gamma_{\min}} + 1 - \alpha_{\min})^{-1/\gamma_{\max}}. \tag{80}$$

Similarly, if l is such that $b_l = b_{\max}$, then

$$1 = \sum_{k=1}^{n} \alpha_k b_k^{\gamma_k} = \alpha_l b_{\max}^{\gamma_l} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_k^{\gamma_k}$$
$$= \alpha_l s^{\gamma_l} b_{\min}^{\gamma_l} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_k^{\gamma_k} \ge \alpha_l s^{\gamma_l} b_{\min}^{\gamma_l} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_{\min}^{\gamma_k}$$
$$\ge \alpha_l s^{\gamma_{\min}} b_{\min}^{\gamma_{\max}} + \sum_{\substack{k=1\\k\ne l}}^{n} \alpha_k b_{\min}^{\gamma_{\max}}$$
$$= b_{\min}^{\gamma_{\max}} (\alpha_l s^{\gamma_{\min}} + 1 - \alpha_l) \ge b_{\min}^{\gamma_{\max}} (\alpha_{\min} s^{\gamma_{\min}} + 1 - \alpha_{\min}) \tag{81}$$

and therefore

$$b_{\min} \le (\alpha_{\min} s^{\gamma_{\min}} + 1 - \alpha_{\min})^{-1/\gamma_{\max}}. \tag{82}$$

Here $\gamma_{\min}$, $\gamma_{\max}$ are the lowest and highest value, respectively, of $\gamma_1, \gamma_2, \ldots, \gamma_n$, and $\alpha_{\min}$ is the lowest value of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Put

$$q(s) = \max((\alpha_{\min} s^{-\gamma_{\min}} + 1 - \alpha_{\min})^{1/\gamma_{\max}^2},$$
$$(\alpha_{\min} s^{\gamma_{\min}} + 1 - \alpha_{\min})^{-1/\gamma_{\max}^2}) \le 1. \tag{83}$$

Combining the inequalities (76), (78), (80), and (82) produces

$$\frac{b_{\max}'}{b_{\min}'} \le \frac{b_{\max}}{b_{\min}} q(s). \tag{84}$$

Now consider the sequence of iterations $b_{\min}^{(0)} = b_{\min}$, $b_{\min}^{(1)} = b_{\min}', \ldots$ and $b_{\max}^{(0)} = b_{\max}$, $b_{\max}^{(1)} = b_{\max}', \ldots$. The series $s^{(j)} = b_{\max}^{(j)}/b_{\min}^{(j)}$, $j=0, 1, \ldots$ is nonincreasing due to the inequality (84) and bounded from below by unity, so it converges to a limit $s^* \ge 1$. Assume that $s^* > 1$. Because $q(s)$ is a decreasing function of s, $q(s^{(j)}) \le q(s^*) < 1$ and the series $s^{(j)}$ decreases at least as fast as a geometric series with quotient $q(s^*)$. In a finite number of terms, it falls below the level s^*. Therefore, the assumption that $s^* > 1$ is false, and $s^{(j)}$ converges to unity. Then $b_1^{(j)}, b_2^{(j)}, \ldots, b_n^{(j)}$ and therefore $a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)}$ converge to unity and from Eq. (58), the sequence of the iterated values $W_k^{(j)}(0)$ converges to $W_k(0)$, $k=1, 2, \ldots, n$.

5. Concluding remarks

This paper provides explicit procedure to obtain the exact solution of equilibrium pricing in a production economy with heterogeneous investors. Each investor maximizes the expected utility from lifetime consumption, taking place at discrete times. Interest rates are determined by economic variables such as the characteristics of the production process, the individual investors' preferences, and the wealth distribution across the participants. Such model provides a tool for quantitative study of the effect of changes in economic conditions on interest rates.

The algorithm is constructive and converges to the equilibrium solution. The convergence is proven for the case of $\gamma_k \ge 1$, $k=1, 2, \ldots, n$, for which the uniqueness of the equilibrium has been established (cf. Karatzas and Shreve, 1998). All other steps of the procedure, however, are valid in general for any positive values of the risk tolerance coefficients. If some of the $\gamma_1, \gamma_2, \ldots, \gamma_n$ are smaller than unity and the values $W_k^{(j)}(0)$ fail to converge to the input values $W_k(0)$, $k=1, 2, \ldots, n$ after a reasonable number of iterations, a search over the space of positive values of $v_1, v_2, \ldots, v_n$ need to be made.

While this paper concentrates on the case that the participants have isoelastic utility functions (4), it can be extended to more general class of utilities. Suppose the k–th investor maximizes the objective (24), where $U_k(C)$ has a positive, decreasing continuous derivative $U_k'(C)$ with $U_k'(0)=\infty$, $U_k'(\infty)=0$, $k=1, 2, \ldots, n$. Denote the inverse of the derivative by $I_k(x)=U_k'^{-1}(x)$. Then the optimal consumption is given by

$$C_{ik} = I_k\left(\frac{Y_i}{\Lambda_k p_{ik}}\right), \tag{85}$$

where Λ_k is a positive constant satisfying the condition

$$W_k(0) = \frac{1}{Y_0} E \sum_{i=1}^{m} Y_i I_k\left(\frac{Y_i}{\Lambda_k p_{ik}}\right) \tag{86}$$

for $k=1, 2, \ldots, n$ (cf. Karatzas and Shreve, 1998, Theorems 3.6.3 and 4.4.5). Put

$$K_i(Y) = \sum_{k=1}^{n} I_k\left(\frac{Y}{\Lambda_k p_{ik}}\right) \quad i=1,2,\ldots,m. \tag{87}$$

Then Eqs. (30), (31), and (33) through (40) still hold. The algorithm consisting of making an initial choice of the constants $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, determining $Y_0, Y_1, \ldots, Y_m$ from Eqs. (39) and (31), setting new values of the constants from Eq. (86), and repeating the calculations may still be applicable, although a proof of convergence is not provided here.

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