DISCONTINUOUS DEMAND FUNCTIONS: ESTIMATION AND PRICING

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Dynamic pricing and learning:

- Learning optimal selling price from accumulating sales data
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- Cont. armed MAB, observing demand $d(p)$ and reward $r(p) = p \cdot d(p)$
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- Learning optimal selling price from accumulating sales data
- Cont. armed MAB, observing demand \( d(p) \) and reward \( r(p) = p \cdot d(p) \)
- Standard assumption: \( d(\cdot) \) is continuous
Nous admettons que la fonction $F(p)$ qui exprime la loi de la demande ou du débit est une fonction continue...
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Motivation

Is the assumption of continuous demand functions reasonable?
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- Price comparison websites: Substantial empirical evidence that seller’s rank heavily influences its demand. Ignoring these discontinuities may distort parameter estimates by 50 to 100 percent. (Baye et al., *J. Econ. Manag. Strategy* 2009)
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- Rankings in online marketplaces (e.g. Amazon’s BuyBox)
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# Motivation

## The Theory and Practice of Revenue Management (International Series in Operations Research & Management Science) (Paperback)

by Peter Wirtz, Kalyan T. Talluri

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- **Price comparison websites**: Substantial empirical evidence that seller’s rank heavily influences its demand. Ignoring these discontinuities may distort parameter estimates by 50 to 100 percent. (Baye et al., *J. Econ. Manag. Strategy* 2009)

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- Product search with price thresholds
Motivation
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- Many online applications challenge Cournot’s continuity assumption
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- Many online applications challenge Cournot’s continuity assumption
- Not treated in dynamic pricing or MAB literature
Central questions

1. Is there a substantial cost of neglecting demand discontinuities in dynamic pricing and learning?
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2. If yes, how to implement estimation and pricing in the presence of demand discontinuities?
Model

- **Price-setting monopolist**: decision variable $p_t \in [p, \bar{p}]$
**Model**

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- **Consumer demand**: Poisson random variable with mean $d(p_t)$

\[
d(p) = \begin{cases} 
  e^{\alpha_0 + \beta_0 p} & \text{if } \kappa_0 \leq p \leq \kappa_1 \\
  e^{\alpha_n + \beta_n p} & \text{if } \kappa_n < p \leq \kappa_{n+1} \quad (n = 1, \ldots, N)
\end{cases}
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- **Model uncertainty:**

  unknown demand parameters \( \theta_n = (\alpha_n, \beta_n) \quad (n = 0, 1, \ldots, N) \)

  unknown discontinuity points \( \kappa_n \quad (n = 1, \ldots, N) \)

  \( \theta = (\theta_0, \theta_1, \ldots, \theta_N) \in \Theta \)

  \( \kappa = (\kappa_1, \ldots, \kappa_N) \in \mathcal{K} \)
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  $\theta = (\theta_0, \theta_1, \ldots, \theta_N) \in \Theta$
  $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathcal{K}$

- **Pricing policy:** $\pi = (p_1, p_2, \ldots)$ non-anticipating
Performance

- Revenue loss in $T$ periods relative to a clairvoyant
Performance

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  - Single-period revenue function $R(p, \kappa, \theta) = p d(p, \kappa, \theta)$
Performance

- Revenue loss in $T$ periods relative to a clairvoyant
  
  - Single-period revenue function $R(p, \kappa, \theta) = p d(p, \kappa, \theta)$
  
  - Regret or “revenue loss due to demand model uncertainty”

\[
\Delta_\pi(T, \kappa, \theta) = \sum_{t=1}^{T} \mathbb{E}_\pi \left\{ \sup_{p \in [p, \bar{p}]} \{ R(p, \kappa, \theta) \} - R(p_t, \kappa, \theta) \right\}
\]
Performance

- Revenue loss in $T$ periods relative to a clairvoyant
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$$

- **Objective:** choose $\pi$ to minimize

$$
\mathcal{R}_\pi(T) = \sup \{ \Delta_\pi(T, \kappa, \theta) : \kappa \in \mathcal{K}, \theta \in \Theta \}
$$
Central questions

1. Is there a substantial cost of neglecting demand discontinuities in dynamic pricing and learning?

2. If yes, how to implement estimation and pricing in the presence of demand discontinuities?
Cost of ignoring demand discontinuities
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No discontinuity

Loss \approx T^{1/2}
Cost of ignoring demand discontinuities

- No discontinuity: Loss $\approx T^{1/2}$
- Ignored discontinuity: Loss $\approx T$
Central questions

1. Is there a substantial cost of neglecting demand discontinuities in dynamic pricing and learning?

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Estimating a discontinuous demand function

- **Two-step maximum likelihood estimation:**
  - Log-likelihood function
    \[
    \mathcal{L}_t : (\kappa, \theta) \mapsto \sum_{s=1}^{t} \sum_{n=0}^{N} \left( d_s \vartheta_n \cdot (1, p_s) - e^{\vartheta_n \cdot (1, p_s)} \right) \mathbb{I}\{\kappa_n < p_s \leq \kappa_{n+1}\}
    \]
    \[
    \hat{\theta}_t(\kappa) = \arg \max_{\theta} \mathcal{L}_t(\kappa, \theta)
    \]

  - Step 1 (discontinuity estimation)
    \[
    \hat{\kappa}_t = \arg \max_{\kappa} \mathcal{L}_t(\hat{\kappa}_t, \hat{\theta}_t(\hat{\kappa}_t))
    \]
  - Step 2 (demand parameter estimation)
    \[
    \hat{\theta}_t = \hat{\theta}_t(\hat{\kappa}_t)
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Estimating a discontinuous demand function

- Two-step maximum likelihood estimation:
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    \]
    \[
    \hat{\vartheta}_t(\kappa) = \arg \max_{\vartheta} \{\mathcal{L}_t(\kappa, \vartheta)\}
    \]
  - Step 1 (discontinuity estimation)
    \[
    \hat{\kappa}_t = \arg \max_{\mathcal{K}} \{\mathcal{L}_t(\mathcal{K}, \hat{\vartheta}_t(\mathcal{K}))\}
    \]
  - Step 2 (demand parameter estimation)
    \[
    \hat{\vartheta}_t = \hat{\vartheta}_t(\hat{\kappa}_t)
    \]
Estimating a discontinuous demand function

\[ D(p) = \exp(a_1 + b_1 p) \]

\[ D(p) = \exp(a_2 + b_2 p) \]
Estimating a discontinuous demand function
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\[ p(2) \leq \hat{\kappa}_1 < p(3) \]
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\[ p(3) \leq \hat{\kappa}_1 < p(4) \]
Estimating a discontinuous demand function

\[ p(4) \leq \hat{\kappa}_1 < p(5) \]
Estimating a discontinuous demand function

\[ p(5) \leq \hat{\kappa}_1 < p(6) \]
Estimating a discontinuous demand function

\[
p(6) \leq \hat{\kappa}_1 < p(7)
\]
Estimating a discontinuous demand function

\[ p(7) \leq \hat{\kappa}_1 < p(8) \]
Estimating a discontinuous demand function

\[ p(8) \leq \hat{\kappa}_1 < p(9) \]
Estimating a discontinuous demand function

Highest likelihood if $p(4) \leq \hat{\kappa}_1 < p(5)$. 
Designing a near-optimal policy

Discontinuity estimation and pricing policy $\pi$

Time horizon $\{1, \ldots, T\}$. 

Designing a near-optimal policy

Discontinuity estimation and pricing policy $\pi$

Time horizon $\{1, \ldots, T\}$.

(1) [Explore] Use $M$ equidistant prices $p = p_1 < \cdots < p_M = \bar{p}$.
Designing a near-optimal policy

Discontinuity estimation and pricing policy $\pi$

Time horizon $\{1, \ldots, T\}$.

1. [Explore] Use $M$ equidistant prices $p = p_1 < \cdots < p_M = \bar{p}$.

2. [Estimate] Compute $\hat{\kappa}$ and $\hat{\theta}$. 
Designing a near-optimal policy

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(1) [Explore] Use $M$ equidistant prices $p = p_1 < \cdots < p_M = \bar{p}$.

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(3) [Exploit] Based on $\hat{\kappa}$ and $\hat{\theta}$, use the estimated optimal price in the remaining $T - M$ periods,
Designing a near-optimal policy

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Time horizon $\{1, \ldots, T\}$.

1. [Explore] Use $M$ equidistant prices $p = p_1 < \cdots < p_M = \bar{p}$.

2. [Estimate] Compute $\hat{\kappa}$ and $\hat{\theta}$.

3. [Exploit] Based on $\hat{\kappa}$ and $\hat{\theta}$, use the estimated optimal price in the remaining $T - M$ periods, but a factor $\log(M)/M$ away from the estimated discontinuities.
Analysis of estimation errors

Theorem (discontinuity estimation error)

There exist constants $M_1, z_1, \gamma_1 > 0$ such that, if $M \geq M_1$, then

$$\mathbb{P}_\pi \left\{ |\hat{\kappa}_n - \kappa_n| > \frac{z_1 \log M}{M} \text{ for some } n = 1, \ldots, N \right\} \leq \frac{\gamma_1}{M}.$$
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Theorem (parameter estimation error)

There exist constants $M_2, z_2, \gamma_2 > 0$ such that, if $M \geq M_2$, then

$$\mathbb{P}_\pi \left\{ \|\hat{\theta}_n - \theta_n\|^2 > \frac{z_2 \log M}{M} \text{ for some } n = 0, 1, \ldots, N \right\} \leq \frac{\gamma_2}{M}.$$
Sufficient condition for good performance

Theorem (upper bound on regret)

There exists a constant $C > 0$ such that, if $M = \lceil \sqrt{T} \rceil$, then

$$R_\pi(T) \leq C \sqrt{T} \log T$$

for all $T \geq 4(N + 1)^2$. 
Summary of results

No discontinuity

Loss \approx T^{1/2}

Ignored discontinuity

Loss \approx T
Summary of results

- Discontinuity estimation:
  - Ignored discontinuity
    - Loss $\approx T$
  - No discontinuity
    - Loss $\approx T^{1/2}$
  - Discontinuity estimation
    - Loss $\approx T^{1/2} \log T$
Some intuition
• What if discontinuities vary over time?
Extensions

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  Include change-point detection module in policy
  Retains $O(\sqrt{T} \log T)$ regret
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Message of this talk

Neglecting discontinuities can cost a lot (linear regret)
Taking it into account retains asymptotic optimality
Extensions in the paper: changing discontinuity points, inventory constraints

Interesting research problems:
- Rank-induced discontinuities in other problems?
- Nonparametric approach to discontinuous MABs.
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